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2010 J. Phys. A: Math. Theor. 43 055205

(http://iopscience.iop.org/1751-8121/43/5/055205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 055205 (31pp) doi:10.1088/1751-8113/43/5/055205

Periodic boundary data for an integrable model of stimulated Raman scattering: long-time asymptotic behavior

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Received 18 September 2009, in final form 30 November 2009
Published 14 January 2010
Online at stacks.iop.org/JPhysA/43/055205

Abstract

We study the long-time asymptotic behavior of the solution to the initial-boundary value (IBV) problem in the quarter plane ($x > 0, t > 0$) for nonlinear integrable equations of stimulated Raman scattering. We consider the case of zero initial condition and periodic boundary data ($p\,e^{i\omega t}$). Using the steepest descent method for oscillatory matrix Riemann–Hilbert problems we show that the solution of the IBV problem has different asymptotic behavior in different regions. The solution takes the form of

- a plane wave of finite amplitude, when $0 < x < \omega_0^2 t$,
- a modulated elliptic wave of finite amplitude, when $\omega_0^2 t < x < \omega^2 t$, and
- a self-similar vanishing (as $t \to \infty$) wave, when $x > \omega^2 t$.

For the IBV problem with nonzero initial condition and the same periodic boundary data, the solution to this problem is qualitatively similar to that of this study with the only difference that the solitons (of finite amplitude) can appear in the region $x > \omega^2 t$.

PACS numbers: 02.30.Jr, 02.30.Ik

1. Introduction

Publications devoted to the phenomenon of stimulated Raman scattering are very numerous and we cite papers close to our considerations. It is well known that stimulated Raman scattering (SRS) is described by three coupled partial differential equations (PDEs). Initial boundary value problems for these equations are well posed [1]. The SRS equations (1) are integrable reductions of them in a special case of the transient limit [2, 3]. In other words, the SRS equations admit the Lax pair and the inverse scattering transform can be applied.

1751-8113/10/055205+31$30.00

Paper [2] is devoted to the Raman soliton generation from laser inputs in the transient SRS model. It was shown that a boundary value problem on the semi-line does induce the generation of solitons by pairs. Besides, this paper provides the derivation of the SRS equations when group velocity dispersion is taken into account. The case of zero group velocity dispersion was studied in [3], where the IBV problem in the finite domain $[0, L] \times [0, T]$ for the transient SRS equations was considered and the long-distance behavior of the system was established via the third Painlevé transcendent. The authors used the method [4] based on the simultaneous spectral analysis of the two parts forming the Lax pair and a matrix Riemann–Hilbert problem on the complex k-plane. These results show that the initial-boundary value (IBV) problem for the SRS equations is a nice model of PDEs, which can be solved by this method without a restriction caused by the so-called global relation between spectral functions [4, 5]. Such a restriction takes place for the most of integrable equations because the method [4] involves more boundary values than in the corresponding well-posed IBV problem. Such an over-determination of the boundary data implies the above-mentioned global relation. In the case of the SRS equations all spectral functions are uniquely defined by given initial and boundary data only. The problem in the finite domain was also considered in [6], where rigorous analysis of the Riemann–Hilbert problem was done.

In the present paper, the IBV problem for the SRS equations is studied in the domain $(x > 0, t > 0)$ with zero initial function and simple periodic boundary data. The similar problem with nonzero initial function, vanishing at infinity, was studied in [7]. In general, one can propose different matrix RH problems suitable for the given IBV problem. In [7], such a matrix Riemann–Hilbert problem was proposed, which provided the existence of the solution for all t and allowed to obtain explicit formula for the asymptotics of the solution. Using the steepest descent method of Deift and Zhou [8] for the oscillatory matrix RH problem, the asymptotics of the solution of the IBV problem was obtained in the form of a self-similar vanishing wave traveling in the region $x > \omega^2 t$ (see section 3.3). In this paper, using ideas of [10, 11], we obtain explicit formulas for the asymptotics of the solution of the IBV problem in the complementary region $0 < x < \omega^2 t$. In the region $\omega_0^2 t < x < \omega^2 t$ with ω_0^2 determined by the parameters of the boundary values (3) and (4):

$$\omega_0^2 = \frac{-8l^3\omega^2}{27 - 18l^2 - l^4 + \sqrt{(1-l^2)(9-l^2)^3}}, \qquad -1 < l < 0,$$

the solution takes the form of a modulated elliptic wave of finite amplitude while in the region $0 < x < \omega_0^2 t$ it takes the form of a plane wave. To make the asymptotic analysis more transparent, we consider the case when the initial function $u(x) \equiv 0$. For nonzero initial function the results are similar.

The IBV problem under consideration is

$$2\mathrm{i}q_t = \mu, \qquad \mu_x = 2\mathrm{i}\nu q, \qquad \nu_x = \mathrm{i}(\bar{q}\mu - q\bar{\mu}), \qquad x \in (0,\infty), \qquad t \in (0,\infty), \tag{1}$$

with initial function

$$q(x,0) = u(x), \qquad x \in (0,\infty), \tag{2}$$

and periodic boundary condition

$$\mu(0,t) = p\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad p > 0, \tag{3}$$

$$\nu(0,t) = l = \mathrm{Const}. \tag{4}$$

We suppose that the function $u(x)$ is of the Schwartz-type function, i.e.

$$u(x) \in \mathcal{C}^\infty(0,\infty), \qquad x^m u_x^{(n)}(x) \in L^\infty(0,\infty), \qquad m, n = 0, 1, 2, \ldots. \tag{5}$$

For definiteness we assume that $p = |\mu(0,t)| > 0$, $\omega > 0$ and $l < 0$. The case $\omega < 0, l > 0$ is realized when we pass to the complex conjugated system (1). Since (1) implies

$$\frac{\partial}{\partial x}(\nu^2(x,t) + |\mu(x,t)|^2) = 0,$$

in what follows we assume that

$$\nu^2(x,t) + |\mu(x,t)|^2 \equiv 1$$

and, particularly, $p^2 + l^2 = 1$. Our considerations are valid if the boundary conditions (3) and (4) are replaced by

$$\mu(0,t) = p\,\mathrm{e}^{\mathrm{i}\omega t} + v(t), \qquad \nu(0,t) = l + w(t), \tag{6}$$

where $v(t)$ and $w(t)$ are given functions of the Schwartz type in $t \in (0,\infty)$. Such an IBV problem was considered in [12], where a generation of asymptotic solitons by boundary data (6) was studied using the Marchenko integral equations.

Note that if $q(x,t)$ is real and $2q = v_x$, $\mu = \mathrm{i}\sin v$, $\nu = \cos v$, then the SRS equations are reduced to the sine-Gordon equation

$$v_{xt} = \sin v. \tag{7}$$

The asymptotic behavior of the rapidly decreasing (as $|x| \to \infty$) solution to the sine-Gordon equation (7) was studied in [13].

2. The matrix Riemann–Hilbert problem and solution of the IBV problem

2.1. Preliminaries

For studying the initial boundary value problem (1)–(4), we will use the Lax pair in the form of the over-determined system of differential equations. They are

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = Q(x,t)\Phi, \tag{8}$$

$$\Phi_t = \frac{\mathrm{i}}{4k}\widehat{Q}(x,t)\Phi, \tag{9}$$

where

$$\begin{gathered}\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad Q(x,t) = \begin{pmatrix}0 & q(x,t)\\ -\bar{q}(x,t) & 0\end{pmatrix},\\ \widehat{Q}(x,t) = \begin{pmatrix}\nu(x,t) & \mathrm{i}\mu(x,t)\\ -\mathrm{i}\bar{\mu}(x,t) & -\nu(x,t)\end{pmatrix},\end{gathered} \tag{10}$$

and $\Phi(x,t,k)$ is a 2×2 matrix-valued function and $k \in \mathbb{C}$ is a parameter. It is easy to verify that the differential equations (8) and (9) are compatible if and only if the entries $q(x,t)$, $\mu(x,t)$, $\nu(x,t)$ of the matrices $Q(x,t)$ and $\widehat{Q}(x,t,k)$ satisfy the SRS equations (1).

To formulate the matrix Riemann–Hilbert problem related to the SRS model, we need to introduce spectral functions defined by the initial function $u(x)$ and the boundary values $\mu(0,t)$ and $\nu(0,t)$. Let the initial function of the Schwartz type $u(x)$ be given and let the vector-function $\Psi(x,k)$ be the solution of the equation

$$\Psi_x + \mathrm{i}k\sigma_3\Psi = \begin{pmatrix}0 & u(x)\\ -\bar{u}(x) & 0\end{pmatrix}\Psi, \qquad 0 < x < \infty,$$

with the boundary condition

$$\lim_{x\to\infty}\Psi(x,k)\,\mathrm{e}^{-\mathrm{i}kx} = \begin{pmatrix}0\\1\end{pmatrix}.$$

Then the vector-function $\Psi(x,k)$ defines the map

$$\mathbb{S}:\{u(x)\}\to\{a(k),b(k)\} \tag{11}$$

by the formula

$$\begin{pmatrix} b(k)\\ a(k)\end{pmatrix} := \Psi(0,k).$$

The functions $a(k)$ and $b(k)$ are called the spectral data corresponding to the initial function $u(x)$. They possess the following properties:

- $a(k)$ and $b(k)$ are analytic in $k\in\mathbb{C}_+$ and smooth in $k\in\mathbb{C}_+\cup\mathbb{R}$ functions represented in the form

$$a(k)=1+\int_0^\infty \alpha(y)\,\mathrm{e}^{\mathrm{i}ky}\,\mathrm{d}y,\qquad b(k)=\int_0^\infty \beta(y)\,\mathrm{e}^{\mathrm{i}ky}\,\mathrm{d}y,$$

  where $\alpha(y),\beta(y)$ are of the Schwartz-type functions (5):
- $|a(k)|^2+|b(k)|^2\equiv 1,\qquad k\in\mathbb{R}$;
- $a(k)=1+O(k^{-1}),\qquad b(k)=O(k^{-1}),\qquad k\to\infty.$

The map (11) is invertible [14].

The boundary values $\mu(0,t)=p\mathrm{e}^{\mathrm{i}\omega t}$ and $\nu(0,t)=l$ $(p^2+l^2=1)$ and the matrix $\widehat{Q}(0,t,k)$ (10) give the equation

$$\Phi_t=\frac{\mathrm{i}}{4k}\begin{pmatrix} l & \mathrm{i}p\,\mathrm{e}^{\mathrm{i}\omega t}\\ -\mathrm{i}p\,\mathrm{e}^{-\mathrm{i}\omega t} & -l\end{pmatrix}\Phi. \tag{12}$$

To define the spectral data corresponding to the boundary values, we choose the solution of (12) in the form

$$\mathcal{E}(t,k)=\frac{1}{2}\,\mathrm{e}^{\mathrm{i}\omega\sigma_3 t/2}\begin{pmatrix} \varkappa(k)+\dfrac{1}{\varkappa(k)} & \varkappa(k)-\dfrac{1}{\varkappa(k)}\\ \varkappa(k)-\dfrac{1}{\varkappa(k)} & \varkappa(k)+\dfrac{1}{\varkappa(k)}\end{pmatrix}\mathrm{e}^{-\mathrm{i}\Omega(k)\sigma_3 t}, \tag{13}$$

where

$$\varkappa(k)=\sqrt[4]{\frac{k-\bar E}{k-E}},\qquad \Omega(k)=\frac{\omega}{2k}\sqrt{(k-E)(k-\bar E)}$$

and

$$E=\frac{l+\mathrm{i}p}{2\omega}=E_1+\mathrm{i}E_2,\qquad \bar E=E_1-\mathrm{i}E_2.$$

To fix branches of the roots, we choose the cut in the complex k-plane along the curve $\hat\gamma$, where $\operatorname{Im}\Omega(k)=0$, and define $\varkappa(k)$ and $\Omega(k)$ in such a way that

$$\varkappa(k)=1+O(k^{-1}),\qquad \Omega(k)=\frac{\omega}{2}+O(k^{-1})\qquad \text{as}\quad k\to\infty.$$

The set $\Sigma:=\{k\in\mathbb{C}|\operatorname{Im}\Omega(k)=0\}$ (figure 1) consists of the real line $\operatorname{Im}k=0$ and the circle arc $\hat\gamma$, which is defined by equations

$$\left(k_1-\frac{|E|^2}{2E_1}\right)^2+k_2^2=\left(\frac{|E|^2}{2E_1}\right)^2,\qquad k_1^2+k_2^2\geqslant|E|^2,\qquad k_1=\operatorname{Re}k,\qquad k_2=\operatorname{Im}k.$$

The real axis divides the circle arc $\hat\gamma$ into two symmetric arcs γ and $\bar\gamma$: $\hat\gamma=\gamma\cup\bar\gamma$.

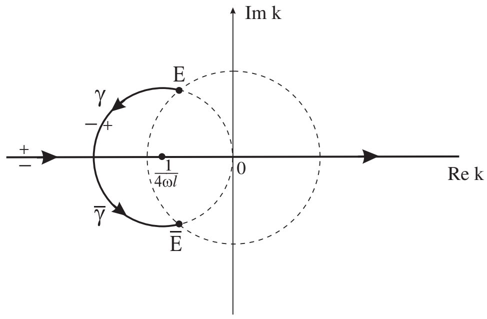

Figure 1. The oriented contour Σ.

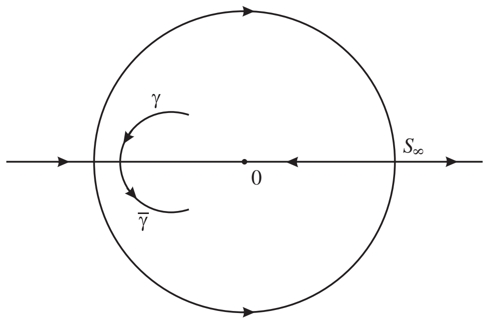

Figure 2. The augmented contour $\hat{\Sigma}$ for the RH_{xt} problem.

Let us define the contour $\Sigma = \mathbb{R} \cup \gamma \cup \bar{\gamma}$ with the orientation as in figure 1. Denoting $\Omega_\pm(k)$, $\varkappa_\pm(k)$ the boundary values of $\Omega(k)$, $\varkappa(k)$ on the cut $\hat{\gamma}$ from the left (+) and right (−) sides of the cut, we have

$$\Omega_+(k) = -\Omega_-(k), \qquad \varkappa_-(k) = \mathrm{i}\varkappa_+(k).$$

The matrix-valued function $\mathcal{E}(t,k)$ in (13) is analytic in $\mathbb{C}\backslash\{\hat{\gamma} \cup \{0\}\}$ and it has an essential singularity at the point 0. The spectral data corresponding to the boundary values $\mu(0,t) = p\,\mathrm{e}^{\mathrm{i}\omega t}$ and $\nu(0,t) = l$ $(p^2 + l^2 = 1)$ are defined as follows:

$$A(k) = \frac{1}{2}\left(\varkappa(k) + \frac{1}{\varkappa(k)}\right), \qquad B(k) = \frac{1}{2}\left(\varkappa(k) - \frac{1}{\varkappa(k)}\right) \qquad k \in \mathbb{C}\setminus\hat{\gamma}. \tag{14}$$

Let $\hat{a}(k) = \bar{a}(\bar{k})A(k) + \bar{b}(\bar{k})B(k)$ and $\hat{b}(k) = a(k)B(k) - b(k)A(k)$ be the auxiliary spectral data. They satisfy the determinant relation

$$\hat{a}(k)\overline{\hat{a}(\bar{k})} + b(k)\overline{\hat{b}(\bar{k})} \equiv 1, \qquad k \in \mathbb{R} \cup \gamma \cup \bar{\gamma}.$$

Define the following function:

$$f(k) := \mathrm{i}[\overline{\hat{a}_-(\bar{k})\hat{a}_+(\bar{k})}]^{-1}, \qquad k \in \gamma,$$

where sign $\pm$ denote boundary values on γ. Then $\hat{a}_-(k)\hat{a}_+(k) = -\mathrm{i}\bar{f}^{-1}(\bar{k})$ for $k \in \bar{\gamma}$.

2.2. The Riemann–Hilbert problem

Let us define the augmented contour $\hat{\Sigma}$ as follows: $\hat{\Sigma} = \mathbb{R} \cup \gamma \cup \bar{\gamma} \cup \mathcal{S}_\infty$ (figure 2). Here the circle $\mathcal{S}_\infty$ has the large enough radius $|\mathcal{S}_\infty|$ that $\hat{a}(k) \neq 0$ for $k > |\mathcal{S}_\infty|$. Then the matrix Riemann–Hilbert problem RH_{xt} introduced in [7] is as follows: find a 2×2 matrix-valued function $M(x, t, k)$ such that

- $M(x, t, k)$ is analytic for $k \in \mathbb{C} \setminus \hat{\Sigma}$;
- $M(x, t, k)$ is bounded in neighborhoods of the end points E and $\bar{E}$;
- $M_-(x, t, k) = M_+(x, t, k) J(x, t, k), \quad k \in \hat{\Sigma} \setminus (E \cup \bar{E})$, where $M_\mp(x, t, k)$ are non-tangential limiting values of $M(x, t, k)$ as k approaches to the contour from the "$\mp$" side of $\hat{\Sigma} \setminus (E \cup \bar{E})$ and

$$J(x,t,k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & |k| < |\mathcal{S}_\infty|, \qquad \operatorname{Im} k = 0; \\ \begin{pmatrix} 0 & -\mathrm{i}\, \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ -\mathrm{i}\, \mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix}, & k \in \gamma \cup \bar{\gamma}; \end{cases}$$

$$J(x,t,k) = \begin{cases} \begin{pmatrix} \hat{a}(k) & 0 \\ \overline{\hat{b}(\bar{k})}\, \mathrm{e}^{2\mathrm{i}t\theta(k)} & \hat{a}^{-1}(k) \end{pmatrix}, & |k| = |\mathcal{S}_\infty|, \qquad \operatorname{Im} k < 0; \\ \begin{pmatrix} 1 & \rho(k)\, \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ \bar{\rho}(k)\, \mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 + |\rho(k)|^2 \end{pmatrix}, & |k| > |\mathcal{S}_\infty|, \qquad \operatorname{Im} k = 0; \\ \begin{pmatrix} \overline{\hat{a}(\bar{k})} & \hat{b}(k)\, \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & \overline{\hat{a}^{-1}(\bar{k})} \end{pmatrix}, & |k| = |\mathcal{S}_\infty|, \qquad \operatorname{Im} k > 0; \end{cases}$$

  with $\rho(k) = \frac{\hat{b}(k)}{\hat{a}(k)}$ and $\theta(k) = \frac{1}{4k} + k\frac{x}{t}$;
- $M(x, t, k) = I + O(k^{-1}), \quad k \to \infty$.

The following theorem is proved in the paper [7].

Theorem 2.1. *Let $u(x)$ be a function of the Schwartz type (5) on the semi-axes, $\nu(0, t) = l$ $(-1 < l < 0)$ and $\mu(0, t) = p\, \mathrm{e}^{\mathrm{i}\omega t}$ $(\omega > 0,\ 0 < p < 1)$. Let $\{a(k), b(k), A(k), B(k)\}$ be the corresponding spectral data and $\{\hat{a}(k), \hat{b}(k)\}$ be the auxiliary spectral data. Then the Riemann–Hilbert problem RH_{xt} has a unique solution $M(x, t, k)$. The functions q(x, t), $\mu(x, t)$ and $\nu(x, t)$ given by the equations*

$$q(x,t) = 2\mathrm{i} \lim_{k\to\infty} [kM(x,t,k)]_{12}, \tag{15}$$

$$\hat{Q}(x,t) = \begin{pmatrix} \nu(x,t) & \mathrm{i}\mu(x,t) \\ -\mathrm{i}\bar{\mu}(x,t) & -\nu(x,t) \end{pmatrix} := -M(x,t,0)\sigma_3 M^{-1}(x,t,0), \tag{16}$$

satisfy the SRS equations (1), the initial condition

$$q(x,0) = u(x), \qquad x \in (0,\infty)$$

and the boundary data

$$\nu(0,t) = l, \qquad \mu(0,t) = p\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad t \in (0,\infty).$$

3. Asymptotic behavior of the solution of the IBV problem

In this section, we study the long-time asymptotic behavior of the solution to the IBV problem (1)–(4). We show that there exist three different asymptotic formulas which describe the long-time behavior of the solution in the three different regions of the first quarter of the xt-plane. To fix ideas of the asymptotic analysis and to make it more transparent we restrict our attention to the special case when the initial function is equal to zero identically. We will use the steepest descent method [8] of Deift and Zhou and its generalization [10, 11] for problems with periodic boundary data. Numerous technical details of this method become much more simple in this special case. In particular, the contour $\hat{\Sigma}$ of the RH problem can be simplified by excluding the circle $\mathcal{S}_\infty$.

For the case $u(x) \equiv 0$, $\mu(0,t) = p\,\mathrm{e}^{\mathrm{i}\omega t}$, $p \in (0,1)$, $\omega > 0$, $\nu(0,t) = l$, $l \in (-1,0)$, the corresponding spectral functions are as follows:

$$a(k) \equiv 1, \qquad b(k) \equiv 0,$$

$$\hat{a}(k) = A(k) = \frac{1}{2}\left(\varkappa(k) + \frac{1}{\varkappa(k)}\right), \qquad \hat{b}(k) = B(k) = \frac{1}{2}\left(\varkappa(k) - \frac{1}{\varkappa(k)}\right), \tag{17}$$

where $\varkappa(k) = \sqrt[4]{\frac{k-\bar{E}}{k-E}}$, $E = \frac{l+\mathrm{i}p}{2\omega}$. These formulas show that spectral data $A(k)$ and $B(k)$ are analytic functions everywhere with exception of the arc $\gamma \cup \bar{\gamma}$ and $A(k) \neq 0$. The reflection coefficient takes the form

$$\rho(k) = B(k)A(k) = -\bar{\rho}(k). \tag{18}$$

We recall that the complex k-plane is cut along the contour $\gamma \cup \bar{\gamma}$. The jump of $\rho(k)$ over $\gamma \cup \bar{\gamma}$ defines a function $f(k)$, i.e.

$$f(k) = \rho_-(k) - \rho_+(k).$$

This formula can be rewritten in the form

$$f(k) = \frac{\mathrm{i}}{A_-(k)A_+(k)} = \frac{1}{B_+(k)A_+(k)} = \frac{2\mathrm{i}}{\operatorname{Im} E} X(k), \tag{19}$$

with $X(k) = \sqrt{(k-E)(k-\bar{E})}$. To fix the branch of the square root we put $X(0) > 0$. The last formula (19) shows that $f(k)$ has analytic continuation on $\mathbb{C} \setminus (\gamma \cup \bar{\gamma})$.

Lemma 3.1. *Let $\hat{a}(k)$, $\hat{b}(k)$ and $\rho(k)$ be given by equations (17) and (18). Then the main Riemann–Hilbert problem RH_{xt} is equivalent to the following one:*

- *matrix valued function $M^{(1)}(x,t,k)$ is analytic in the domain $\mathbb{C} \setminus \Sigma$;*
- *$M_-^{(1)}(x,t,k) = M_+^{(1)}(x,t,k) J^{(1)}(x,t,k)$, $\quad k \in \Sigma = \mathbb{R} \setminus \{0\} \cup \gamma \cup \bar{\gamma}$, where*

$$J^{(1)}(x,t,k) = \begin{cases} \begin{pmatrix} 1 & \rho(k)\,\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ -\rho(k)\,\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1-\rho^2(k) \end{pmatrix}, & k \in \mathbb{R} \setminus \{0\}, \\ \begin{pmatrix} 1 & 0 \\ f(k)\,\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}, & k \in \gamma; \\ \begin{pmatrix} 1 & f(k)\,\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}, & k \in \bar{\gamma}, \end{cases}$$

  where $f(k) = \rho_-(k) - \rho_+(k)$;
- *$M^{(1)}(x,t,k)$ is bounded in neighborhoods of the points E, $\bar{E}$, $k = 0$;*
- *$M^{(1)}(x,t,k) = I + O(k^{-1})$, $\quad k \to \infty$.*

  The functions $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$ are determined by $M^{(1)}(x,t,k)$ in the same way as (15) and (16), where matrix $M^{(1)}(x,t,0)$ is the non-tangential limit of $M^{(1)}(x,t,k)$ as $k \to 0$.

Proof. Since $\hat{a}(k) \equiv A(k) \neq 0$ for all k, the $\mathrm{RH_{xt}}$ problem can be simplified. Indeed, let us transform initial matrix $M(x,t,k)$ to the following one:

$$M^{(1)}(x,t,k) = M(x,t,k)G^{(1)}(x,t,k),$$

where $G^{(1)}(x,t,k) = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ for $|k| > |\mathcal{S}_\infty|$ and

$$G^{(1)}(x,t,k) = \begin{cases} \begin{pmatrix} A^{-1}(k) & -B(k)\,\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & A(k) \end{pmatrix}, & |k| < |\mathcal{S}_\infty|, \qquad \operatorname{Im} k > 0, \\ \begin{pmatrix} A(k) & 0 \\ -B(k)\,\mathrm{e}^{2\mathrm{i}t\theta(k)} & A^{-1}(k) \end{pmatrix}, & |k| < |\mathcal{S}_\infty|, \qquad \operatorname{Im} k < 0. \end{cases}$$

This transformation eliminates the circle $\mathcal{S}_\infty$, where the jump matrix $J(x,t,k) = G^{(1)}(x,t,k)$. It is easy to see that the matrix-valued function $M^{(1)}(x,t,k)$ is analytic in the domains $\mathbb{C}\setminus\Sigma$, bounded at the branch points E, $\bar{E}$ and at the origin $k = 0$. The new jump matrix coincides with the jump matrix $J^{(1)}(x,t,k)$. Furthermore, since $M_-(x,t,0) = M_+(x,t,0)$ and

$$G^{(1)}(x,t,k) = \begin{cases} \begin{pmatrix} A^{-1}(k) & O(\mathrm{e}^{-t\operatorname{Im}k/2|k|^2}) \\ 0 & A(k) \end{pmatrix}, & k \to 0, \qquad \operatorname{Im} k > 0, \\ \begin{pmatrix} A(k) & 0 \\ O(\mathrm{e}^{t\operatorname{Im}k/2|k|^2}) & A^{-1}(k) \end{pmatrix}, & k \to 0, \qquad \operatorname{Im} k < 0 \end{cases}$$

becomes diagonal in the non-tangential limit as $k \to 0$, we have

$$M_-^{(1)}(x,t,0) = M_+^{(1)}(x,t,0)\big[G_+^{(1)}(0)\big]^{-1}G_-^{(1)}(0) = M_+^{(1)}(x,t,0)A^{2\sigma_3}(0).$$

Therefore $\nu(x,t)$ and $\mu(x,t)$, given by $M_\pm^{(1)}(x,t,0)$ according to (16) ($\hat{Q}(x,t) = -M_\pm^{(1)}(x,t,0)\sigma_3[M_\pm^{(1)}(x,t,0)]^{-1}$), coincide with the ones given by $M(x,t,0)$. Finally, since $M^{(1)}(x,t,k) = M(x,t,k)$ for $|k| > |\mathcal{S}_\infty|$, the function $q(x,t)$ is the same as that in (15). □

The contour $\hat{\gamma} = \gamma \cup \bar{\gamma}$ plays a crucial role in the description of the long-time asymptotics in the region $x < \omega^2 t$, where the asymptotics has a finite order, while this contour can be eliminated in the region $x > \omega^2 t$ where asymptotics takes the form of a self-similar vanishing wave (see [7] and subsection 3.3 of this paper).

3.1. Plane wave asymptotics $\left(\xi_0 < \xi = \sqrt{\frac{t}{4x}} < \infty\right)$

The jump matrices $J^{(1)}(x,t,k)$ depend on $\exp\{\pm 2\mathrm{i}t\theta(k)\}$. Therefore the signature table of the imaginary part of the phase function $\theta(k) = \theta(k,\xi)$ plays a very important role. The phase function $\theta(k,\xi)$ has real stationary points $\pm\xi$ ($\xi^2 := t/4x$) and its imaginary part is as follows:

$$\operatorname{Im}\theta(k) = \frac{|k|^2 - \xi^2}{4|k|^2\xi^2}\operatorname{Im} k.$$

Thus $\operatorname{Im}\theta(k) > 0$ ($\operatorname{Im}\theta(k) < 0$) for k lying in the lower (upper) half-disk and out of the upper (lower) half-disk defined by the circle $|k|^2 = \xi^2$ (figure 3). For all $\xi \in [\xi_0, \infty)$, where ξ_0 is given below in (24), $\operatorname{Im}\theta(k,\xi)$ is negative along all γ and positive along all $\bar{\gamma}$. Therefore the jump matrices $J^{(1)}(x,t,k)$ are unbounded (in t) on the contour $\hat{\gamma} = \gamma \cup \bar{\gamma}$. Hence for $\xi \in [\xi_0, \infty)$ we have to follow the modification of the nonlinear steepest descent method as suggested in [10, 11, 15, 16] and find a new phase function $g(k) = g(k,\xi)$, instead of the function $\theta(k,\xi)$, which transforms the original Riemann–Hilbert problem to the model RH problem of the finite-gap type. Such a g-function does really exist. It leads to the finite-gap model problems of zero genus for $\xi \in [\xi_0, \infty)$ and genus 1 for $\xi \in \left(\frac{1}{2\omega}, \xi_0\right)$. They are explicitly

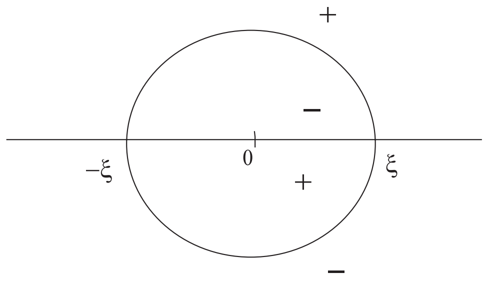

Figure 3. The signature table of the function $\operatorname{Im}\theta(k)$.

solved by using elementary functions in the first region and the elliptic theta functions in the second region, respectively.

In the region $\xi \in [\xi_0, \infty)$, we shall use the following g-function:

$$g(k) = \Omega(k) + 14\xi^2 X(k) = \left(\frac{\omega}{2k} + \frac{1}{4\xi^2}\right) X(k), \qquad k \neq 0, \tag{20}$$

where $X(k) = \sqrt{(k-E)(k-\bar{E})} = \sqrt{(k-E_1)^2 + E_2^2}$ and $E_1 = l/2\omega$, $E_2 = p/2\omega$. This function has the asymptotic behavior similar to the phase function $\theta(k)$, i.e.

$$g(k) = \begin{cases} \frac{1}{4k} + g_0(\xi) + O(k), & k \to 0, \\ \frac{k}{4\xi^2} + g_\infty(\xi) + O(k^{-1}), & k \to \infty, \end{cases}$$

where

$$g_0(\xi) = \frac{1}{8\omega\xi^2} - \frac{l\omega}{2}, \qquad g_\infty(\xi) = \frac{\omega}{2} - \frac{l}{8\omega\xi^2}. \tag{21}$$

The differential of this function is equal to

$$\mathrm{d}g(k) = \frac{k^3 - l2\omega k^2 + l\xi^2 k - \xi^2 2\omega}{4\xi^2 k^2 X(k)}\,\mathrm{d}k.$$

On the other hand it can be written as follows:

$$\mathrm{d}g(k) = \frac{(k-\lambda_-)(k-\lambda)(k-\lambda_+)}{4\xi^2 k^2 X(k)}\,\mathrm{d}k.$$

Comparing the two forms of the differential dg, we find the system of equations

$$\begin{cases} \lambda + \lambda_- + \lambda_+ = \dfrac{l}{2\omega}, \\ \lambda(\lambda_- + \lambda_+) + \lambda_-\lambda_+ = \dfrac{l}{\xi^2}, \\ \lambda\lambda_-\lambda_+ = \dfrac{\xi^2}{2\omega}, \end{cases} \tag{22}$$

which define the zeros λ, λ_-, λ_+ as functions on ξ. These zeroes satisfy the equation $Q(k) := k^3 - \frac{l}{2\omega}k^2 + l\xi^2 k - \frac{\xi^2}{2\omega} = 0$ and, therefore, their locations can be easily obtained by comparing of two part of the polynomial $Q(k)$. Namely, the zeros λ, λ_-, λ_+ are the points of the intersection of some cubic curve and a strait line, i.e.

$$k^2\left(k - \frac{l}{2\omega}\right) = -l\xi^2 k + \frac{\xi^2}{2\omega}. \tag{23}$$

A simple analysis of (23) shows that for the case under consideration ($l < 0$ and $\omega > 0$) the following inequalities take place:

$$\lambda_-(\xi) \leqslant \lambda(\xi) < \frac{1}{2l\omega} < 0 < \lambda_+(\xi).$$

Rewrite equation (23) in the equivalent form:

$$(k + \xi\sqrt{-l})k(k - \xi\sqrt{-l}) = \frac{l}{2\omega}\left(k + \frac{\xi}{\sqrt{-l}}\right)\left(k - \frac{\xi}{\sqrt{-l}}\right).$$

This form of equation (23) yields the following inequalities (together with previous ones):

$$-\xi\sqrt{-l} < \lambda_-(\xi) \leqslant \lambda(\xi) < \frac{1}{2l\omega} < 0 < \xi\sqrt{-l} < \lambda_+(\xi) < \frac{\xi}{\sqrt{-l}}.$$

It can be shown that

$$\lambda_-(\xi) \to -\infty, \qquad \lambda_+(\xi) \to +\infty, \qquad \lambda(\xi) \to \frac{1}{2l\omega},$$

as $\xi \to +\infty$. The boundary value ξ_0 of ξ is determined by the equation $\lambda_-(\xi_0) = \lambda(\xi_0)$. Then equations (22) reduce to

$$\begin{aligned}
&\lambda_+(\xi_0) = \frac{l}{2\omega} - 2\lambda(\xi_0),\\
&2\lambda(\xi_0)\lambda_+(\xi_0) = l\xi_0^2 - \lambda^2(\xi_0),\\
&\xi_0^2 = 2\omega\lambda^2(\xi_0)\lambda_+(\xi_0),
\end{aligned}$$

which give the following equation for $\lambda(\xi_0)$:

$$4l\omega\lambda^2(\xi_0) - (3 + l^2)\lambda(\xi_0) + \frac{l}{\omega} = 0.$$

Thus we have

$$\lambda(\xi_0) = \lambda_-(\xi_0) = \frac{3 + l^2 + \sqrt{(1 - l^2)(9 - l^2)}}{8l\omega},$$

$$\lambda_+(\xi_0) = -\frac{3 - l^2 + \sqrt{(1 - l^2)(9 - l^2)}}{4l\omega}$$

and

$$\xi_0^2 = -\frac{27 - 18l^2 - l^4 + \sqrt{(1 - l^2)(9 - l^2)^3}}{32l^3\omega^2}, \tag{24}$$

where $-1 < l < 0$.

In what follows a signature table of the function $\operatorname{Im} g(k) = \operatorname{Im} g(k, \xi)$ plays a very important role. Borderlines between different domains are described by equations

$$\begin{aligned}
&k_2 = 0;\\
&\frac{k_1^2 + k_2^2 + 2\omega\xi^2 k_1}{2\omega\xi^2 k_2} - \frac{2\omega\xi^2 k_2}{k_1^2 + k_2^2 + 2\omega\xi^2 k_1} = \frac{\left(k_1 - \frac{l}{2\omega}\right)^2 - k_2^2 + \frac{1-l^2}{4\omega^2}}{\left(k_1 - \frac{l}{2\omega}\right)k_2},\\
&k_1^2 + k_2^2 > 1/4\omega^2,
\end{aligned}$$

which are equivalent to $\operatorname{Im} g(k) = 0$. The signature table of the function $\operatorname{Im} g(k)$ can be obtained by using for example 'MAPLE' and it is qualitatively depicted in figure 4 for $\xi_0 < \xi < \infty$ and in figure 5 for $\xi = \xi_0$. We denote the large closed contour in these pictures by the deformed oval Γ.

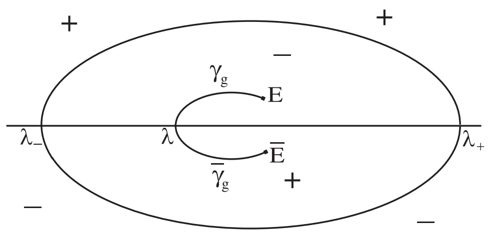

Figure 4. The signature table of the function $\operatorname{Im} g(k)$ ($\xi_0 < \xi < \infty$).

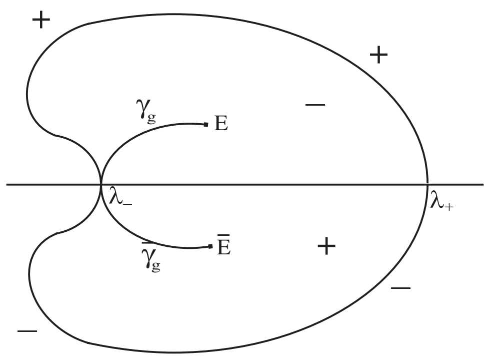

Figure 5. The signature table of the function $\operatorname{Im} g(k)$ ($\xi = \xi_0$).

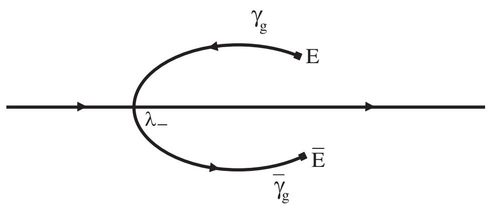

Figure 6. The contour $\Sigma^{(2)}$ for the $M^{(2)}(x, t, k)$-problem.

The Riemann–Hilbert problem for the matrix $M^{(1)}(x, t, k)$ with the jump contour $\Sigma^{(1)} := \mathbb{R} \cup \gamma \cup \bar{\gamma}$ has to be considered with the new phase function $g(k)$ (20) and on a new contour: $\Sigma^{(2)} = \mathbb{R} \cup \gamma_g \cup \bar{\gamma}_g$, where $\operatorname{Im} g(k) = 0$ (see figure 6). More precisely, we put

$$M^{(1)}(x, t, k) = \mathrm{e}^{\mathrm{i}tg_\infty(\xi)\sigma_3} M^{(2)}(x, t, k)\, \mathrm{e}^{\mathrm{i}t[\theta(k)-g(k)]\sigma_3},$$

where the phase function $g(k) = g(k, \xi)$ is defined in (20). Then the matrix $M^{(2)}(x, t, k)$ satisfies the following RH problem:

$$M_-^{(2)}(x, t, k) = M_+^{(2)}(x, t, k) J^{(2)}(x, t, k)$$

with the jump matrix $J^{(2)}(x, t, k)$:

$$\begin{pmatrix} 1 & \rho(k)\, \mathrm{e}^{-2\mathrm{i}tg(k)} \\ -\rho(k)\, \mathrm{e}^{2\mathrm{i}tg(k)} & 1 - \rho^2(k) \end{pmatrix}, \qquad k \in \mathbb{R} \setminus \{0\},$$

$$\begin{pmatrix} e^{-it(g_+(k)-g_-(k))} & 0 \\ f(k)\, e^{it(g_+(k)+g_-(k))} & e^{it(g_+(k)-g_-(k))} \end{pmatrix}, \qquad k \in \gamma_g,$$

$$\begin{pmatrix} e^{-it(g_+(k)-g_-(k))} & f(k)\, e^{-it(g_+(k)+g_-(k))} \\ 0 & e^{it(g_+(k)-g_-(k))} \end{pmatrix}, \qquad k \in \bar{\gamma}_g.$$

To make the off-diagonal entries of the jump matrices to be independent of t we have to put

$$g_-(k) + g_+(k) = 0, \qquad k \in \gamma_g \cup \bar{\gamma}_g. \tag{25}$$

Chosen above the g-function satisfies this condition.

To obtain suitable factorization of the jump matrix on $\mathbb{R}$ let us perform the δ-transformation:

$$M^{(3)}(x,t,k) = M^{(2)}(x,t,k)\delta^{-\sigma_3}(k),$$

where the function $\delta(k)$ has the form

$$\delta(k) = \exp\left\{\frac{1}{2\pi i}\int_{\lambda_-(\xi)}^{\lambda_+(\xi)} \frac{\log(1-\rho^2(s))\, ds}{s-k}\right\}, \qquad k \in C \setminus [\lambda_-(\xi), \lambda_+(\xi)] \tag{26}$$

and $\lambda_\pm(\xi)$ are the stationary points of the phase function $g(k)$. Then the jump matrix $J^{(3)}(x,t,k)$ has a lower/upper factorization for $k \in [\lambda_-(\xi), \lambda_+(\xi)]$ and an upper/lower factorization for $k \notin [\lambda_-(\xi), \lambda_+(\xi)]$:

$$J^{(3)}(x,t,k) = \begin{pmatrix} 1 & A(k)B(k)\delta_+^2(k)\, e^{-2itg(k)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A(k)B(k)\delta_-^{-2}(k)\, e^{2itg(k)} & 1 \end{pmatrix},$$

$$= \begin{pmatrix} 1 & 0 \\ -\rho(k)\delta^{-2}(k)\, e^{2itg(k)} & 1 \end{pmatrix} \begin{pmatrix} 1 & \rho(k)\delta^2(k)\, e^{-2itg(k)} \\ 0 & 1 \end{pmatrix},$$

where we use the identity

$$\frac{\rho(k)}{1-\rho^2(k)} = A(k)B(k).$$

Due to equation (25), the jump matrix takes the form

$$J^{(3)}(x,t,k) = \begin{pmatrix} e^{-2itg_+(k)} & 0 \\ f(k)\delta^{-2}(k) & e^{2itg_+(k)} \end{pmatrix}, \qquad k \in \gamma_g$$

$$= \begin{pmatrix} e^{-2itg_+(k)} & -\bar{f}(\bar{k})\delta^2(k) \\ 0 & e^{2itg_+(k)} \end{pmatrix}, \qquad k \in \bar{\gamma}_g$$

on the contour $\gamma_g \cup \bar{\gamma}_g$. The jump contour $\Sigma^{(3)}$ for $M^{(3)}(x,t,k)$-problem is the previous one, i.e. $\Sigma^{(3)} := \Sigma^{(2)}$. Let us define a decomposition of the complex k-plane into eight domains $D_1, \ldots, D_8$ separated by their common boundary $\Sigma^{(4)}$ as it is shown in figure 7. The contours L_2 and L_5 lie strongly inside of a domain bounded by the deformed oval Γ (dotted line); the contours L_1, L_6 (L_3, L_4) range from the point $\lambda_+(\xi)$ ($\lambda_-(\xi)$) to infinity along the rays $\arg k = \pm\pi/4$ ($\arg k = \pi \mp \pi/4$). Then the next transformation is

$$M^{(4)}(x,t,k) = M^{(3)}(x,t,k)G^{(2)}(k),$$

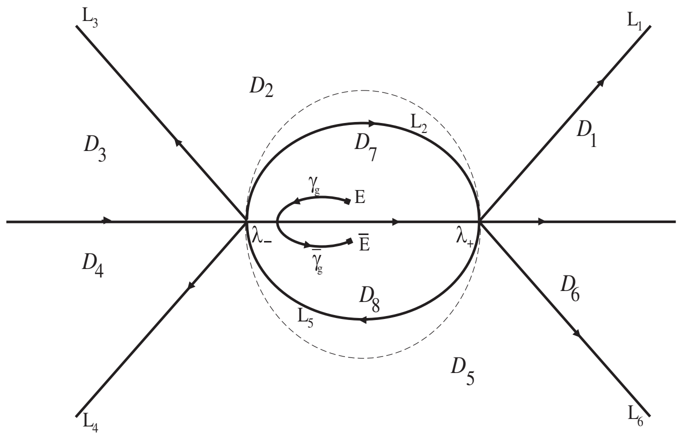

Figure 7. The contour $\Sigma^{(4)}$ for the $M^{(4)}(x,t,k)$-problem.

where

$$G^{(2)}(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\rho(k)\delta^{-2}(k)\,\mathrm{e}^{2\mathrm{i}tg(k)} & 1 \end{pmatrix}, & k \in D_1 \cup D_3, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & k \in D_2 \cup D_5, \\ \begin{pmatrix} 1 & -\rho(k)\delta^{2}(k)\,\mathrm{e}^{-2\mathrm{i}tg(k)} \\ 0 & 1 \end{pmatrix}, & k \in D_4 \cup D_6, \end{cases} \tag{27}$$

$$G^{(2)}(k) = \begin{cases} \begin{pmatrix} 1 & A(k)B(k)\delta^{2}(k)\,\mathrm{e}^{-2\mathrm{i}tg(k)} \\ 0 & 1 \end{pmatrix}, & k \in D_7, \\ \begin{pmatrix} 1 & 0 \\ A(k)B(k)\delta^{-2}(k)\,\mathrm{e}^{2\mathrm{i}tg(k)} & 1 \end{pmatrix}, & k \in D_8. \end{cases} \tag{28}$$

The $G^{(2)}$-transformation leads to the following RH problem:

$$M_-^{(4)}(x,t,k) = M_+^{(4)}(x,t,k)J^{(4)}(x,t,k)$$

on the contour $\Sigma^{(4)}$ depicted in figure 7 with jump matrices $J^{(4)}(x,t,k)$ which are equal to the identity matrix on the real axis; they coincide with matrices $G^{(2)}(k)$ from (27)–(28) written for the contours $k \in L_j$ $(j = 1, 2, \ldots, 6)$. It is easy to see that $J^{(4)}(x,t,k) = I + O(\mathrm{e}^{-\epsilon t})$ as $t \to \infty$ and $k \in L_j$ with exception of some neighborhoods of the stationary points $\lambda_\pm$. Therefore we have to pay the main attention to the contour $\gamma_g \cup \bar{\gamma}_g$, where the jump matrices take the form

$$\begin{aligned} J^{(4)}(x,t,k) &= G_+^{(2)^{-1}}(k)J^{(3)}(x,t,k)G_-^{(2)}(k) \\ &= \begin{pmatrix} 1 & -A_+(k)B_+(k)\delta^2(k)\,\mathrm{e}^{-2\mathrm{i}tg_+(k)} \\ 0 & 1 \end{pmatrix} J^{(3)}(x,t,k) \begin{pmatrix} 1 & A_-(k)B_-(k)\delta^2(k)\,\mathrm{e}^{2\mathrm{i}tg_+(k)} \\ 0 & 1 \end{pmatrix}, \\ &\quad k \in \gamma_g, \\ &= \begin{pmatrix} 1 & 0 \\ -A_+(k)B_+(k)\delta^{-2}(k)\,\mathrm{e}^{2\mathrm{i}tg_+(k)} & 1 \end{pmatrix} J^{(3)}(x,t,k) \begin{pmatrix} 1 & 0 \\ A_-(k)B_-(k)\delta^{-2}(k)\,\mathrm{e}^{-2\mathrm{i}tg_+(k)} & 1 \end{pmatrix}, \\ &\quad k \in \bar{\gamma}_g. \end{aligned}$$

Using equalities $1 - f(k)A_+(k)B_+(k) = 0$, $1 + \bar{f}(\bar{k})A_+(k)B_+(k) = 0$ and $A_+(k)B_+(k) = -A_-(k)B_-(k)$, which follow from the definition of the function $f(k)$, we obtain

$$J^{(4)}(x,t,k) = \begin{cases} \begin{pmatrix} 0 & -f^{-1}(k)\delta^2(k) \\ f(k)\delta^{-2}(k) & 0 \end{pmatrix}, & k \in \gamma_g \\ \begin{pmatrix} 0 & f(k)\delta^2(k) \\ -f^{-1}(k)\delta^{-2}(k) & 0 \end{pmatrix}, & k \in \bar{\gamma}_g. \end{cases}$$

Further we would like to obtain the RH problem with a jump matrix, which is independent of k. To do so let us use the factorization

$$J^{(4)}(x,t,k) = \begin{pmatrix} F_+^{-1}(k) & 0 \\ 0 & F_+(k) \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} F_-(k) & 0 \\ 0 & F_-^{-1}(k) \end{pmatrix}$$

which takes place if $F_-(k)F_+(k) = -\mathrm{i}f(k)\delta^{-2}(k)$ for $k \in \gamma_g$ and $F_-(k)F_+(k) = \mathrm{i}f^{-1}(k)\delta^{-2}(k)$ for $k \in \bar{\gamma}_g$.

Thus we come to a scalar Riemann–Hilbert problem: find a scalar function $F(k)$ such that

- $F(k)$ is analytic outside the contour $\gamma_g \cup \bar{\gamma}_g$ oriented downward (from E to $\bar{E}$);
- $F(k)$ does not vanish;
- $F(k)$ satisfies the jump relation:

$$F_-(k)F_+(k) = h(k)\delta^{-2}(k), \qquad k \in \gamma_g \cup \bar{\gamma}_g,$$

where

$$h(k) = \begin{cases} -\mathrm{i}f(k) = A_-^{-1}(k)A_+^{-1}(k) & k \in \gamma_g \\ \mathrm{i}f^{-1}(k) = A_-(k)A_+(k), & k \in \bar{\gamma}_g \end{cases} \qquad \text{and}$$

- $F(k)$ is bounded at infinity.

To solve this RH problem, let us put

$$H(k) = \begin{cases} F(k)A(k), & k \in \mathbb{C}_+ \setminus \gamma_g \\ \dfrac{F(k)}{A(k)}, & k \in \mathbb{C}_- \setminus \bar{\gamma}_g \end{cases}$$

and use the function $X(k) = \sqrt{(k-E)(k-\bar{E})}$. Since

$$\left[\frac{\log H(k)}{X(k)}\right]_+ - \left[\frac{\log H(k)}{X(k)}\right]_- = \frac{\log \delta^{-2}(k)}{X_+(k)}, \qquad k \in \gamma_g \cup \bar{\gamma}_g,$$

and

$$\left[\frac{\log H(k)}{X(k)}\right]_+ - \left[\frac{\log H(k)}{X(k)}\right]_- = \frac{\log A^2(k)}{X(k)}, \qquad k \in \mathbb{R},$$

we have

$$H(k) = \exp\left\{\frac{X(k)}{2\pi\mathrm{i}}\left[\int_{\gamma_g \cup \bar{\gamma}_g} \frac{\log \delta^{-2}(s,\xi)}{s-k}\frac{\mathrm{d}s}{X_+(s)} + \int_{\mathbb{R}} \frac{\log A^2(s)}{s-k}\frac{\mathrm{d}s}{X(s)}\right]\right\}.$$

The functions $H(k) = H(k,\xi)$ and $F(k) = F(k,\xi)$ are bounded at infinity. Indeed, $F(\infty,\xi) = H(\infty,\xi) = \exp(\mathrm{i}\phi(\xi))$, where

$$\phi(\xi) = \frac{1}{2\pi}\left[\int_{\gamma_g \cup \bar{\gamma}_g} \log \delta^{-2}(s,\xi)\frac{\mathrm{d}s}{X_+(s)} + \int_{\mathbb{R}} \log A^2(s)\frac{\mathrm{d}s}{X(s)}\right].$$

Using (26), $1 - \rho^2(k) = A^{-2}(k)$ and

$$\frac{1}{\pi \mathrm{i}} \int_{\gamma_g \cup \bar{\gamma}_g} \frac{\mathrm{d}k}{(s-k)X_+(k)} = -\frac{1}{X(s)},$$

we obtain

$$\phi(\xi) = \frac{1}{2\pi} \left(\int_{-\infty}^{\lambda_-(\xi)} + \int_{\lambda_+(\xi)}^{\infty} \right) \log A^2(k) \frac{\mathrm{d}k}{X(k)}. \tag{29}$$

Now, since

$$J^{(4)}(x,t,k) = F_+^{-\sigma_3}(k) J^{(\mathrm{mod})} F_-^{\sigma_3}(k), \qquad k \in \gamma_g \cup \bar{\gamma}_g,$$

where

$$J^{(\mathrm{mod})} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

the next step is as follows:

$$M^{(5)}(x,t,k) = F^{\sigma_3}(\infty) M^{(4)}(x,t,k) F^{-\sigma_3}(k).$$

Then we have

$$M_-^{(5)}(x,t,k) = M_+^{(5)}(x,t,k) J^{(5)}(x,t,k), \qquad k \in \Sigma^{(4)},$$

where

$$J^{(5)}(x,t,k) = \begin{cases} I & k \in \mathbb{R}, \\ J^{(\mathrm{mod})} & k \in \gamma_g \cup \bar{\gamma}_g, \\ I + \mathrm{O}(e^{-\varepsilon t}) & k \in L_j, \qquad j = 1, 2, \ldots, 6. \end{cases}$$

The analysis of the parametrix solutions near the end points E, $\bar{E}$ and the stationary points $\lambda_-, \lambda, \lambda_+$ are very similar to the analysis done in [15] and [9], respectively. In the first case, since the local representation of $g(k)$ at the points E and $\bar{E}$ is characterized by a square-root-type behavior:

$$g(k) \sim g_0(E,\xi)\sqrt{k-E}, \quad k \to E; \qquad g(k) \sim \bar{g}_0(\bar{E},\xi)\sqrt{k-\bar{E}}, \quad k \to \bar{E},$$

the relevant model Riemann–Hilbert problems are solvable in terms of Bessel functions while in the second case of the real stationary points they are solvable in terms of parabolic cylinder functions. Skipping the technical details, we have the following asymptotic representation of the function $M^{(5)}(x,t,k)$:

$$M^{(5)}(x,t,k) = \left(I + O\left(\frac{1}{t^{1/2}} \right) \right) M^{(\mathrm{mod})}(x,t,k),$$

where $M^{(\mathrm{mod})}(x,t,k)$ solves the zero-gap model problem (cf [11]):

$$M_-^{(\mathrm{mod})}(x,t,k) = M_+^{(\mathrm{mod})}(x,t,k) J^{(\mathrm{mod})} \qquad k \in \gamma_g \cup \bar{\gamma}_g.$$

with the constant jump matrix

$$J^{(\mathrm{mod})} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

It is worth mentioning that the error order $t^{-1/2}$ comes from the contribution of the stationary phase points $\lambda_\pm(\xi)$.

To solve the model problem let us use the function

$$\varkappa(k) = \left(\frac{k-\bar{E}}{k-E} \right)^{1/4}$$

introduced in the first section. Since $\varkappa_-(k) = \mathrm{i}\varkappa_+(k)$ on the cut $\gamma_g \cup \bar{\gamma}_g$ the explicit solution takes the form

$$M^{(\mathrm{mod})}(x,t,k) = \frac{1}{2}\begin{pmatrix} \varkappa(k) + \dfrac{1}{\varkappa(k)} & \varkappa(k) - \dfrac{1}{\varkappa(k)} \\ \varkappa(k) - \dfrac{1}{\varkappa(k)} & \varkappa(k) + \dfrac{1}{\varkappa(k)} \end{pmatrix}.$$

Finally we have the following chain of transformations of the RH problem:

$$\begin{aligned}
&M(x,t,k) = M^{(1)}(x,t,k)[G^{(1)}(k)]^{-1}\\
&M^{(1)}(x,t,k) = \mathrm{e}^{\mathrm{i}tg_\infty(\xi)\sigma_3}M^{(2)}(x,t,k)\,\mathrm{e}^{\mathrm{i}t[\theta(k)-g(k)]\sigma_3},\\
&M^{(2)}(x,t,k) = M^{(3)}(x,t,k)\delta^{\sigma_3}(k),\\
&M^{(3)}(x,t,k) = M^{(4)}(x,t,k)[G^{(2)}(k)]^{-1}\\
&M^{(4)}(x,t,k) = F^{-\sigma_3}(\infty)M^{(5)}(x,t,k)F^{\sigma_3}(k),\\
&M^{(5)}(x,t,k) = M^{(\mathrm{mod})}(x,t,k)(I + O(t^{-1/2})).
\end{aligned}$$

Let us emphasize that any matrix $M^{(j)}(x,t,k)$ $(j = 1, 2, \ldots)$ defines the same functions $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$ since all bordering matrices are diagonal at the point $k = \infty$ and $k = 0$. By theorem 2.1, $q(x,t) = 2\mathrm{i}m_{12}(x,t)$. Take into account the chain of our transformations and using the equality $m_{12}^{(\mathrm{mod})}(x,t) = \mathrm{i}E_2/2 = \mathrm{i}p/4\omega$ we have

$$\begin{aligned}
q(x,t) &= 2\mathrm{i}m_{12}(x,t) = 2\mathrm{i}m_{12}^{(1)}(x,t) = 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}tg_\infty(\xi)}m_{12}^{(2)}(x,t)\\
&= 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}tg_\infty(\xi)}m_{12}^{(3)}(x,t) + \mathrm{O}(t^{-1/2})\\
&= 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}tg_\infty(\xi)}m_{12}^{(4)}(x,t) + \mathrm{O}(t^{-1/2})\\
&= 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}tg_\infty(\xi)}m_{12}^{(5)}(x,t)F^{-2}(\infty) + \mathrm{O}(t^{-1/2})\\
&= 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}tg_\infty(\xi)}m_{12}^{(\mathrm{mod})}(x,t)F^{-2}(\infty) + \mathrm{O}(t^{-1/2})\\
&= -\frac{p}{2\omega}\exp\left[2\mathrm{i}tg_\infty(\xi) - 2\mathrm{i}\phi(\xi)\right] + \mathrm{O}(t^{-1/2}).
\end{aligned}$$

Again by theorem 2.1, $\hat{Q}(x,t) = -M(x,t,0)\sigma_3 M^{-1}(x,t,0)$. Hence

$$\begin{aligned}
\hat{Q}(x,t) &= \begin{pmatrix} \nu(x,t) & \mathrm{i}\mu(x,t) \\ -\mathrm{i}\mu(x,t) & -\nu(x,t) \end{pmatrix}\\
&= -\mathrm{e}^{\mathrm{i}tg_\infty(\xi)\sigma_3}F^{-\sigma_3}(\infty)M^{(\mathrm{mod})}(t,\xi,0)\sigma_3\,(M^{(\mathrm{mod})}(t,\xi,0))^{-1}F^{\sigma_3}(\infty)\,\mathrm{e}^{-\mathrm{i}tg_\infty(\xi)\sigma_3}\\
&\quad + O(t^{-1/2}).
\end{aligned}$$

Since

$$M^{(\mathrm{mod})}(x,t,0) = \frac{1}{|E|}\begin{pmatrix} E_1 & \mathrm{i}E_2 \\ -\mathrm{i}E_2 & -E_1 \end{pmatrix}$$

and $F(\infty) = \mathrm{e}^{\mathrm{i}\phi(\xi)}$, we find

$$\nu(x,t) = l + \mathrm{O}(t^{-1/2}), \qquad \mu = p\exp\left[2\mathrm{i}tg_\infty(\xi) - 2\mathrm{i}\phi(\xi)\right] + \mathrm{O}(t^{-1/2}),$$

where $g_\infty(\xi)$ are given in (21) and $\phi(\xi)$ in (29).

Theorem 3.1. *The solution of the IBV problem (1)–(4) for $t \to \infty$ in the region $0 < x < \omega_0^2 t$ takes the form of the plane wave*

$$\begin{aligned}
q(x,t) &= -\frac{p}{2\omega}\exp\left[\mathrm{i}\omega t - \mathrm{i}\frac{l}{\omega}x - 2\mathrm{i}\phi(\xi)\right] + O(t^{-1/2}),\\
\mu(x,t) &= p\exp\left[\mathrm{i}\omega t - \mathrm{i}\frac{l}{\omega}x - 2\mathrm{i}\phi(\xi)\right] + O(t^{-1/2}),\\
\nu(x,t) &= l + O(t^{-1/2}),
\end{aligned}$$

where

$$\phi(\xi) = \frac{1}{2\pi}\left(\int_{-\infty}^{\lambda_-(\xi)} + \int_{\lambda_+(\xi)}^{\infty}\right)\log A^2(k)\frac{\mathrm{d}k}{X(k)},$$

where $\lambda_\pm(\xi)$ *are the stationary points of the function* $g(k,\xi)$ $(\xi_0 \leqslant \xi \leqslant \infty)$*, and* $A(k)$ *is defined by (17).*

Remark 3.1. If $x = 0$ $(\xi = \infty)$, then $\lambda_-(\infty) = -\infty$ and $\lambda_+(\infty) = +\infty$. Hence

$$\phi(\infty) = 0.$$

Therefore the plane wave asymptotics of the solution matches the boundary conditions at $x = 0$.

3.2. Modulated elliptic asymptotics $\omega_0^2 t < x < \omega^2 t$ $\left(\omega_0^2 = 1/4\xi_0^2\right)$

When $\xi = \xi_0$ zeroes $\lambda_-(\xi_0)$ and $\lambda(\xi_0)$ (24) of the differential $\mathrm{d}g$ are equal (see figure 5) and for $\xi < \xi_0$ they become complex conjugated, i.e.,

$$\mathrm{d}g(k) = \frac{(k-\lambda(\xi))(k-\bar{\lambda}(\xi))(k-\lambda_+(\xi))}{4\xi^2k^2\sqrt{(k-E)(k-\bar{E})}}\,\mathrm{d}k.$$

As a result the previous consideration fails. We need to introduce a new g-function for this region. Let us fix the contour of the RH problem putting $\xi = \xi_0$, i.e.

$$\Sigma^{(0)} = \mathbb{R}\cup\gamma_0\cup\bar{\gamma}_0,$$

where $\gamma_0 := \gamma_{g(\xi_0)}$ and $\bar{\gamma}_0 := \bar{\gamma}_{g(\xi_0)}$.

A suitable g-function for $\xi < \xi_0$ can be obtained as follows. First, we need to introduce a new real stationary point $\lambda_-(\xi)$ which must be a zero of the new differential $\mathrm{d}\hat{g}$. On the other hand, we have to preserve the asymptotic behavior of g-function for large k. To do so we take the differential $\mathrm{d}\hat{g}$ in the form

$$\mathrm{d}\hat{g}(k) = \frac{(k-\lambda_-(\xi))(k-\lambda(\xi))(k-\bar{\lambda}(\xi))(k-\lambda_+(\xi))}{4\xi^2k^2\sqrt{(k-d(\xi))(k-\bar{d}(\xi))(k-E)(k-\bar{E})}}\,\mathrm{d}k, \tag{30}$$

where $\lambda_-(\xi) < \lambda_+(\xi)$ and $\lambda(\xi) = \lambda_1(\xi) + \mathrm{i}\lambda_2(\xi)$ and $d(\xi) = d_1(\xi) + \mathrm{i}d_2(\xi)$ are complex valued functions on ξ. All the functions $\lambda_\pm(\xi)$, $\lambda(\xi)$, $d(\xi)$ have to be determined. Note, if $d(\xi) = \bar{d}(\xi) = \lambda(\xi) = \bar{\lambda}(\xi)$ for $\xi = \xi_0$, then $\mathrm{d}\hat{g}(k,\xi_0) = \mathrm{d}g(k,\xi_0)$ and, hence, the function $\hat{g}(k)$ coincides (up to a constant) with function $g(k)$ from the previous subsection. If $d(\xi) = \lambda(\xi) = E$ and $\lambda_+(\xi) = -\lambda_-(\xi) = |E|$ for $\xi = |E|$, then the function $\hat{g}(k) = \hat{g}(k,|E|)$ coincides (up to a constant) with the phase function $\theta(k)$, i.e.

$$\hat{g}(k,|E|) + \frac{E_1}{2|E|^2} = \theta(k,|E|), \qquad \theta(k,\xi) = \frac{1}{4k} + \frac{k}{4\xi^2},$$

which serves for the problem in the region $\xi < |E|$.

One can see that $\mathrm{d}\hat{g}$ is an Abelian differential of the second kind with poles at the points $(\infty^\pm)$ and $(0^\pm)$ of the Riemann surface (algebraic curve) given by equation

$$w(k) = \sqrt{(k-E)(k-\bar{E})(k-d(\xi))(k-\bar{d}(\xi))}, \qquad E = E_1 + \mathrm{i}E_2 = (l+\mathrm{i}p)/2\omega.$$

We will use the realization of this algebraic curve as a two-sheet Riemann surface. The upper (+) and lower (−) sheet of the surface are two complex planes merged along the cuts γ_d, joining E and $d(\xi)$, and $\bar{\gamma}_d$, joining $\bar{d}(\xi)$ and $\bar{E}$. The points $0^\pm$ and $\infty^\pm$ on the surface are images of the points 0 and ∞ on the complex plane. The branch of the square root is fixed by asymptotics on the upper sheet

$$w(k) = k^2[1 + O(k^{-1})], \qquad k \to \infty^+.$$

The basis $\{\boldsymbol{a}, \boldsymbol{b}\}$ of cycles of this Riemann surface is as follows. The $\boldsymbol{a}$-cycle starts on the upper sheet from the left side of the cut γ_d, goes to the left side of the cut $\bar{\gamma}_d$, proceeds to the lower sheet and then returns to the starting point. The $\boldsymbol{b}$-cycle is a closed clockwise oriented simple loop around the cut γ_d.

We write the Abelian differential $\mathrm{d}\hat{g}(k)$ in the form

$$\mathrm{d}\hat{g}(k) = \frac{k^4 + c_3(\xi)k^3 + c_2(\xi)k^2 + c_1(\xi)k + c_0(\xi)}{4\xi^2 k^2 w(k)}\,\mathrm{d}k \tag{31}$$

and normalize it so that its $\boldsymbol{a}$-period vanishes. Since

$$\int_{\boldsymbol{a}} \mathrm{d}\hat{g} = -2\int_{\bar{d}}^{d} \mathrm{d}\hat{g}(k),$$

where the path of integration is the line segment $[\bar{d}, d]$, this normalization condition means

$$c_2(\xi) = -\frac{\int_{\bar{d}}^{d} \frac{k^4 + c_3(\xi)k^3 + c_1(\xi)k + c_0(\xi)}{k^2 w(k)}\,\mathrm{d}k}{\int_{\bar{d}}^{d} \frac{\mathrm{d}k}{w(k)}}. \tag{32}$$

We require that the function $\hat{g}(k)$ has no logarithmic singularities and behaves for large and small k as the phase $\theta(k) = \frac{1}{4k} + \frac{k}{4\xi^2}$, i.e.

$$\hat{g}(k) = \frac{k}{4\xi^2} + O(1), \quad k \to \infty; \qquad \hat{g}(k) = \frac{1}{4k} + O(1), \quad k \to 0.$$

These requirements imply

$$c_3 = E_1 - d_1(\xi), \qquad c_0 = -\xi^2|E||d(\xi)|, \qquad c_1 = -c_0(E_1/|E|^2 + d_1(\xi)/|d(\xi)|^2).$$

Then we put $\hat{g}(k)$ as a sum of two Abelian integrals of the second kind:

$$\hat{g}(k) = \left(\int_E^k + \int_{\bar{E}}^k\right) \frac{z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0}{8\xi^2 z^2 w(z)}\,\mathrm{d}z. \tag{33}$$

It has a real $\boldsymbol{b}$-period:

$$B_g = \left(\int_E^d + \int_{\bar{E}}^{\bar{d}}\right) \frac{z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0}{8\xi^2 z^2 w(z)}\,\mathrm{d}z.$$

Indeed, taking into account the relation $\int_{\bar{d}}^{d} \mathrm{d}\hat{g} = 0$ and the absence of the residues of $\mathrm{d}\hat{g}$ at the points $\infty^\pm$ and $0^\pm$ we see that in fact $\int_{\bar{E}}^{E} \mathrm{d}\hat{g} = 0$ as well, and that the function $\hat{g}(k)$ can be written as a single Abelian integral

$$\hat{g}(k) = \int_E^k \frac{z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0}{4\xi^2 z^2 w(z)}\,\mathrm{d}z,$$

and, simultaneously, we have indeed

$$B_g = \int_{\boldsymbol{b}} \mathrm{d}g.$$

In what follows, we consider $\hat{g}(k)$ on the upper sheet of the Riemann surface, i.e. on the complex plane with a cut along the contour $\gamma_d \cup \bar{\gamma}_d \cup [d, \lambda_-] \cup [\lambda_-, \bar{d}]$. The integration paths in all integrals are chosen so that they do not intersect the above contour. Then, due to equality $\int_{\bar{E}}^{E} \mathrm{d}\hat{g} = 0$, $\hat{g}(k)$ is a single valued analytic function outside of the above-mentioned cut. It is easy to find the asymptotics of $\hat{g}(k)$ at $k = \infty^{+}$:

$$\hat{g}(k) = \frac{k}{4\xi^2} + \hat{g}_{\infty}(\xi) + O(k^{-1}), \qquad k \to \infty^{+},$$

where

$$\hat{g}_{\infty}(\xi) = \left(\int_{E}^{\infty} + \int_{\bar{E}}^{\infty}\right)\left[\frac{z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0}{8\xi^2 z^2 w(z)} - \frac{1}{8\xi^2}\right]\mathrm{d}z - \frac{l}{8\omega\xi^2} \tag{34}$$

is a real function of ξ. At the point $k = 0^{+}$, the function $\hat{g}(k)$ has the following asymptotics:

$$\hat{g}(k) = \frac{1}{4k} + \hat{g}_0(\xi) + O(k), \qquad k \to 0^{+},$$

where

$$\hat{g}_0(\xi) = \left(\int_{E}^{0^+} + \int_{\bar{E}}^{0^+}\right)\left[\frac{z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0}{8\xi^2 z^2 w(z)} - \frac{1}{8z^2}\right]\mathrm{d}z - \frac{\omega l}{2}$$

is also a real function of ξ. A natural local parameter at the point d is $(k-d)^{1/2}$. Then the local expansion of $\hat{g}(k)$ has the form

$$\hat{g}(k) = B_g + g_1(k-d)^{1/2} + g_2(k-d)^{3/2} + g_3(k-d)^{5/2} + \dots,$$

where B_g is real. The desirable signature table of the function $\operatorname{Im}\hat{g}(k)$ requires three branches of $\operatorname{Im}\hat{g}(k) = 0$ going out from the point d. It is possible if and only if $g_1 = 0$, i.e.

$$(k-d)^{1/2} g'(k)|_{k=d} = \frac{(d-\lambda_-(\xi))(d-\lambda(\xi))(d-\bar{\lambda}(\xi))(d-\lambda_+(\xi))}{\sqrt{(d-E)(d-\bar{E})(d-\bar{d})}} = 0.$$

Since we suppose $\lambda_{\pm}(\xi)$ to be real, $\lambda(\xi) \equiv d(\xi)$ and we have finally that the differential (30) takes the form

$$\mathrm{d}\hat{g}(k) = \frac{(k-\lambda_-(\xi))(k-\lambda_+(\xi))}{4\xi^2 k^2}\sqrt{\frac{(k-d(\xi))(k-\bar{d}(\xi))}{(k-E)(k-\bar{E})}}\,\mathrm{d}k. \tag{35}$$

Thus we have

(i) In view of harmonicity of the function $\operatorname{Im}\hat{g}(k)$ and $\operatorname{Im}\hat{g}(E) = \operatorname{Im}\hat{g}(d) = 0$ the points E and $d = d(\xi)$ are connected by the zero-level curve γ_d (and by the symmetry the points $\bar{E}$ and $\bar{d} = \bar{d}(\xi)$ are connected by the zero-level curve $\bar{\gamma}_d$), where $\operatorname{Im}\hat{g}(k) = 0$.

(ii) Since $\hat{g}(k)$ is real on the real axis, there exist a real point $\lambda_-(\xi)$ and some curve γ_λ connecting λ_- and d (and by the symmetry the points λ_- and $\bar{d}$ are connected by a curve $\bar{\gamma}_\lambda$), where $\operatorname{Im}\hat{g}(k) = 0$.

(iii) Since $\mathrm{d}\hat{g}(k)$ has two real stationary points $\lambda_+(\xi)$ and $\lambda_-(\xi)$ and $\hat{g}(k)$ is real on the real axis, there exist some curve Γ_d connecting d and $\lambda_+(\xi)$ (and by the symmetry the points $\bar{d}$ and $\lambda_+(\xi)$ are connected by a curve $\bar{\Gamma}_d$), where $\operatorname{Im}\hat{g}(k) = 0$. Thus we have a closed contour $\Gamma_d \cup \bar{\Gamma}_d \cup \gamma_\lambda \cup \bar{\gamma}_\lambda$, which we denote again as a deformed oval Γ. These contours and signature table of the function $\operatorname{Im}\hat{g}(k) = 0$ are depicted in figure 8.

In order to define λ_-, λ_+, $d = d_1 + \mathrm{i}d_2$ and $\bar{d} = d_1 - \mathrm{i}d_2$ as functions on ξ, let us compare the representation (31) of the differential $\mathrm{d}\hat{g}$ with its form (35). Then we obtain

$$\begin{cases} \lambda_- + \lambda_+ + 2d_1 = -c_3 = E_1 + d_1, \\ \lambda_-\lambda_+ + 2(\lambda_- + \lambda_+)d_1 + |d|^2 = c_2, \\ 2\lambda_-\lambda_+ d_1 + (\lambda_- + \lambda_+)|d|^2 = -c_1 = -\xi^2\left(E_1\frac{|d|}{|E|} + \frac{d_1}{|d|}|E|\right) \\ \lambda_-\lambda_+|d|^2 = c_0 = -\xi^2|E||d|, \end{cases} \tag{36}$$

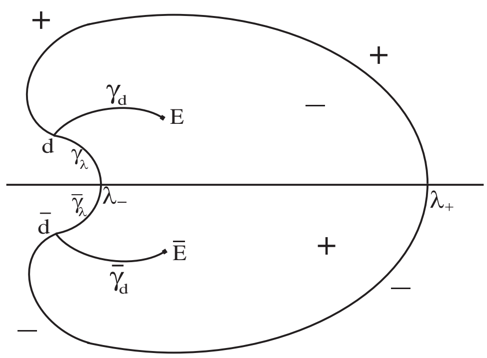

Figure 8. Signature table of Im $\hat{g}(k)$ and deformed oval Γ.

where $c_2(\xi)$ is given by (32). These four (real) equations define four unknown (real) values $\lambda_-(\xi), \lambda_+(\xi), d_1(\xi), d_2(\xi)$. It is easy to see that these equations can be reduced to the pair of nonlinear equations with respect to $\lambda_-(\xi)$ and $\lambda_+(\xi)$. These equations contain a complete elliptic integrals. Indeed, from the first equation above, we have

$$d_1 = E_1 - \lambda_- - \lambda_+ \tag{37}$$

and then, due to the fourth equation,

$$d_2 = \sqrt{\frac{\xi^4|E|^2}{\lambda_-^2\lambda_+^2} - (E_1 - \lambda_- - \lambda_+)^2}. \tag{38}$$

These equations describe the deformation on the parameter ξ of the branch point $d = d_1(\xi) + \mathrm{i}d_2(\xi)$. Together with the third equation (36), they give the following dependence of the stationary points:

$$\lambda_- = G(\lambda_+, \lambda_-, \xi), \qquad G(\lambda_+, \lambda_-, \xi) = -\lambda_+ + E_1\frac{1 - \xi^4\lambda_-^{-2}\lambda_+^{-2}}{1 - \xi^4|E|^2\lambda_-^{-3}\lambda_+^{-3}}. \tag{39}$$

The second equation (36) (the differential $\mathrm{d}\hat{g}$ has zero $\boldsymbol{a}$-period) and (35) gives one more relation on the stationary points:

$$\lambda_+ = H(\lambda_+, \lambda_-, \xi), \qquad H(\lambda_+, \lambda_-, \xi) = \frac{I_0(\lambda_-, d_1, d_2)}{I_1(\lambda_-, d_1, d_2)}, \tag{40}$$

where for $j = 0, 1$

$$I_j(\lambda_-, d_1, d_2) = \int_{d_1 - \mathrm{i}d_2}^{d_1 + \mathrm{i}d_2} \left(1 - \frac{\lambda_-}{z}\right)\sqrt{\frac{(z-d_1)^2 + d_2^2}{(z-E_1)^2 + E_2^2}}\frac{\mathrm{d}z}{z^j}$$
$$= \mathrm{i}(d_2)^2\int_{-1}^{1}\left(1 - \frac{\lambda_-}{d_1 + \mathrm{i}sd_2}\right)\sqrt{\frac{1 - s^2}{(d_1 - E_1 + \mathrm{i}sd_2)^2 + E_2^2}}\frac{\mathrm{d}s}{(d_1 + \mathrm{i}sd_2)^j},$$

and functions $d_1 = d_1(\lambda_+, \lambda_-)$, $d_2 = d_2(\lambda_+, \lambda_-)$ are the same as above (37) and (38). Equations (39) and (40) define the deformations of the stationary points $\lambda_+ = \lambda_+(\xi)$ and $\lambda_- = \lambda_-(\xi)$.

We shall take now an advantage of analyticity of the Riemann–Hilbert data $A(k)$, $B(k)$ and $\rho(k)$ and deform the contour $\gamma_0 \cup \bar\gamma_0$ to the contour $\gamma_d \cup \bar\gamma_d \cup [d, \lambda_-] \cup [\lambda_-, \bar d]$. The part $\gamma_d \cup \bar\gamma_d$ of this contour is chosen in such a way that Im $\hat{g}(k) = 0$ while a line $[d, \lambda_-]$ lies in the domain where $\operatorname{Im}\hat{g}(k) > 0$, and a line $[\lambda_-, \bar{d}]$ lies in the domain where $\operatorname{Im}\hat{g}(k) < 0$. All functions, that had jumps across $\gamma_0 \cup \bar{\gamma}_0$, have now jumps across $\gamma_d \cup \bar{\gamma}_d \cup [d, \lambda_-] \cup [\lambda_-, \bar{d}]$ with the same jump relations as they had before the deformation. In turn, the function $\hat{g}(k)$ has the following properties:

$$\begin{aligned}&\hat{g}_+(k) + \hat{g}_-(k) = 0, && k \in \gamma_d \cup \bar{\gamma}_d;\\ &\hat{g}_+(k) - \hat{g}_-(k) = B_g, && \operatorname{Im} B_g = 0, \quad k \in [d, \lambda_-] \cup [\lambda_-, \bar{d}].\end{aligned}$$

The Riemann–Hilbert problem for the matrix $M^{(1)}(x, t, k)$ has to be considered now on a new contour: $\Sigma^{(1)} := \mathbb{R} \cup \gamma_d \cup \bar{\gamma}_d \cup [d, \lambda_-] \cup [\lambda_-, \bar{d}]$ and with new phase function. More precisely, we put

$$M^{(1)}(x, t, k) = \mathrm{e}^{\mathrm{i}t\hat{g}_\infty(\xi)\sigma_3} M^{(2)}(x, t, k)\, \mathrm{e}^{\mathrm{i}t[\theta(k)-\hat{g}(k)]\sigma_3},$$

where the phase function $\hat{g}(k) = \hat{g}(k, \xi)$ is defined in (33). Then the matrix $M^{(2)}(x, t, k)$ satisfies the following RH problem:

$$M_-^{(2)}(x, t, k) = M_+^{(2)}(x, t, k) J^{(2)}(x, t, k), \qquad k \in \Sigma^{(2)} := \Sigma^{(1)}$$

with the jump matrix $J^{(2)}(x, t, k) = \mathrm{e}^{\mathrm{i}t[\theta(k)-\hat{g}_+(k)]\sigma_3} J^{(1)}(x, t, k)\, \mathrm{e}^{-\mathrm{i}t[\theta(k)-\hat{g}_-(k)]\sigma_3}$, i.e.

$$\begin{aligned}J^{(2)}(x, t, k) &= \begin{pmatrix} 1 & \rho(k)\,\mathrm{e}^{-2\mathrm{i}t\hat{g}(k)} \\ -\rho(k)\,\mathrm{e}^{2\mathrm{i}t\hat{g}(k)} & 1-\rho^2(k)\end{pmatrix}, && k \in \mathbb{R}\setminus\{0\},\\
&= \begin{pmatrix} \mathrm{e}^{-2\mathrm{i}t\hat{g}_+(k)} & 0 \\ f(k) & \mathrm{e}^{2\mathrm{i}t\hat{g}_+(k)}\end{pmatrix}, && k \in \gamma_d,\\
&= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}tB_g} & 0 \\ f(k)\,\mathrm{e}^{\mathrm{i}t(\hat{g}_+(k)+\hat{g}_-(k))} & \mathrm{e}^{\mathrm{i}tB_g}\end{pmatrix}, && k \in [d, \lambda_-],\\
&= \begin{pmatrix} \mathrm{e}^{-2\mathrm{i}t\hat{g}_+(k)} & -\bar{f}(\bar{k}) \\ 0 & \mathrm{e}^{2\mathrm{i}t\hat{g}_+(k)}\end{pmatrix}, && k \in \bar{\gamma}_d,\\
&= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}tB_g} & -\bar{f}(\bar{k})\,\mathrm{e}^{-\mathrm{i}t(\hat{g}_+(k)+\hat{g}_-(k))} \\ 0 & \mathrm{e}^{\mathrm{i}tB_g}\end{pmatrix}, && k \in [\lambda_-, \bar{d}].\end{aligned}$$

Let us perform the δ-transformation

$$M^{(3)}(x, t, k) = M^{(2)}(x, t, k)\delta^{-\sigma_3}(k),$$

where

$$\delta(k) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\lambda_-(\xi)}^{\lambda_+(\xi)} \frac{\log(1-\rho^2(s))\,\mathrm{d}s}{s-k}\right\}, \qquad k \in \mathbb{C}\setminus[\lambda_-(\xi), \lambda_+(\xi)],$$

and $\lambda_\pm(\xi)$ are the stationary points of the phase function $\hat{g}(k) = \hat{g}(k, \xi)$. Then the jump matrix $J^{(3)}(x, t, k)$ with the jump contour $\Sigma^{(3)} := \Sigma^{(2)}$ (see figure 6) has a lower/upper factorization for $k \in [\lambda_-(\xi), \lambda_+(\xi)]$ and an upper/lower factorization for $k \notin [\lambda_-(\xi), \lambda_+(\xi)]$:

$$\begin{aligned}J^{(3)}(x, t, k) &= \begin{pmatrix} 1 & A(k)B(k)\delta_+^2(k)\,\mathrm{e}^{-2\mathrm{i}t\hat{g}(k)} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -A(k)B(k)\delta_-^{-2}(k)\,\mathrm{e}^{2\mathrm{i}t\hat{g}(k)} & 1\end{pmatrix},\\
&= \begin{pmatrix} 1 & 0 \\ -\rho(k)\delta^{-2}(k)\,\mathrm{e}^{2\mathrm{i}t\hat{g}(k)} & 1\end{pmatrix}\begin{pmatrix} 1 & \rho(k)\delta^2(k)\,\mathrm{e}^{-2\mathrm{i}t\hat{g}(k)} \\ 0 & 1\end{pmatrix},\end{aligned}$$

where, again, we use the identity

$$\frac{\rho(k)}{1-\rho^2(k)} = A(k)B(k).$$

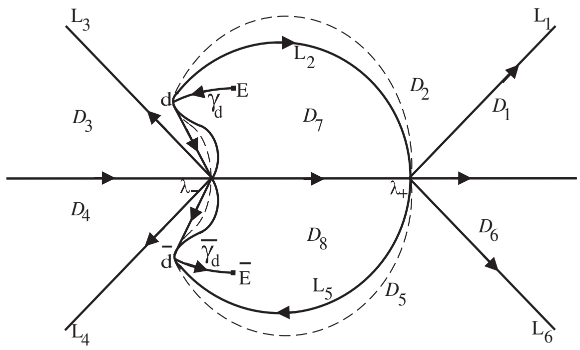

Figure 9. The contour $\Sigma^{(4)}$ for the $M^{(4)}(x,t,k)$-problem.

The jump matrices on the contours $\gamma_d \cup \bar{\gamma}_d \cup [d,\lambda_-] \cup [\lambda_-,\bar{d}]$ are

$$J^{(3)}(x,t,k) = \begin{pmatrix} \mathrm{e}^{-2\mathrm{i}t\hat{g}_+(k)} & 0 \\ f(k)\delta^{-2}(k) & \mathrm{e}^{2\mathrm{i}t\hat{g}_+(k)} \end{pmatrix}, \qquad k \in \gamma_d,$$

$$= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}tB_g} & 0 \\ f(k)\delta^{-2}(k)\,\mathrm{e}^{\mathrm{i}t(\hat{g}_+(k)+\hat{g}_-(k))} & \mathrm{e}^{\mathrm{i}tB_g} \end{pmatrix}, \qquad k \in [d,\lambda_-],$$

$$= \begin{pmatrix} \mathrm{e}^{-2\mathrm{i}t\hat{g}_+(k)} & -\bar{f}(\bar{k})\delta^2(k) \\ 0 & \mathrm{e}^{2\mathrm{i}t\hat{g}_+(k)} \end{pmatrix}, \qquad k \in \bar{\gamma}_d,$$

$$= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}tB_g} & -\bar{f}(\bar{k})\delta^2(k)\,\mathrm{e}^{-\mathrm{i}t(\hat{g}_+(k)+\hat{g}_-(k))} \\ 0 & \mathrm{e}^{\mathrm{i}tB_g} \end{pmatrix}, \qquad k \in [\lambda_-,\bar{d}].$$

Let us define a decomposition of the complex k-plane into eight domains $D_1,\ldots,D_8$ separated by their common boundary $\Sigma^{(4)}$ as it is shown in figure 9. The contours L_2 and L_5 pass through the points λ_-, d, λ_+, $\bar{d}$ and lie strongly in the domain bounded by the deformed oval Γ (see also figure 8) in such a way that domain $D_7 \cup D_8$ contains the contours $\gamma_d \cup \bar{\gamma}_d$; the broken line $[d,\lambda_-] \cup [\lambda_-,\bar{d}]$ lies in the domains where $\operatorname{Im}\hat{g}(k) > 0$ and $\operatorname{Im}\hat{g}(k) < 0$, respectively; the contours L_1, L_6 (L_3, L_4) range from the point $\lambda_+(\xi)$ ($\lambda_-(\xi)$) to infinity along the rays $\arg k = \pm\pi/4$ ($\arg k = \pi \mp \pi/4$). Then the next transformation is

$$M^{(4)}(x,t,k) = M^{(3)}(x,t,k)G^{(2)}(k),$$

where $G^{(2)}(k)$ is given by equations (27) and (28). This transformation leads to the following RH problem:

$$M_-^{(4)}(x,t,k) = M_+^{(4)}(x,t,k)J^{(4)}(x,t,k), \qquad k \in \Sigma^{(4)}$$

on the contour depicted in figure 9 with the jump matrix $J^{(4)}(x,t,k)$ which is equal to the identity matrix on the real axis. $J^{(4)}(x,t,k)$ coincides with matrices $G^{(2)}(k)$ from (27) to (28) written for the contours $k \in L_j$ $(j = 1,2,\ldots,6)$. It is easy to see that $J^{(4)}(x,t,k) = I + O(\mathrm{e}^{-\epsilon t})$ as $t \to \infty$ and $k \in L_j$ with the exception of some neighborhoods of the stationary points $\lambda_\pm$. Furthermore, the jump matrices $J^{(4)}(x,t,k)$ on the broken line $[d,\lambda_-] \cup [\lambda_-,\bar{d}]$ take the form

$$J^{(4)}(x,t,k) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}tB_g} & 0 \\ f(k)\delta^{-2}(k)\,\mathrm{e}^{\mathrm{i}t(\hat{g}_+(k)+\hat{g}_-(k))} & \mathrm{e}^{\mathrm{i}tB_g} \end{pmatrix}, \qquad k \in [d,\lambda_-],$$

$$= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}tB_g} & f(k)\delta^2(k)\,\mathrm{e}^{-\mathrm{i}t(\hat{g}_+(k)+\hat{g}_-(k))} \\ & \mathrm{e}^{\mathrm{i}tB_g} \end{pmatrix}, \qquad k \in [\lambda_-,\bar{d}].$$

Therefore, away from the points d, λ_- and $\bar{d}$, they are closed to the diagonal matrix

$$J_0^{(\mathrm{mod})} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}tB_g(\xi)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}tB_g(\xi)} \end{pmatrix} \tag{41}$$

up to $\mathrm{O}(\mathrm{e}^{-\epsilon t})$ as $t \to \infty$. The jump matrix which has to be factorized is the restriction of the matrix $J^{(4)}(x,t,k)$ on the arcs γ_d and $\bar{\gamma}_d$:

$$\begin{aligned} J^{(4)}(x,t,k) &= G_+^{(2)^{-1}}(k)J^{(3)}(x,t,k)G_-^{(2)}(k) \\ &= \begin{pmatrix} 1 & -A_+(k)B_+(k)\delta^2(k)\,\mathrm{e}^{-2\mathrm{i}t\hat{g}_+(k)} \\ 0 & 1 \end{pmatrix} J^{(3)}(x,t,k) \begin{pmatrix} 1 & A_-(k)B_-(k)\delta^2(k)\,\mathrm{e}^{2\mathrm{i}t\hat{g}_+(k)} \\ 0 & 1 \end{pmatrix}, \\ &\qquad k \in \gamma_d, \\ &= \begin{pmatrix} 1 & 0 \\ -A_+(k)B_+(k)\delta^{-2}(k)\,\mathrm{e}^{2\mathrm{i}t\hat{g}_+(k)} & 1 \end{pmatrix} J^{(3)}(x,t,k) \begin{pmatrix} 1 & 0 \\ A_-(k)B_-(k)\delta^{-2}(k)\,\mathrm{e}^{-2\mathrm{i}t\hat{g}_+(k)} & 1 \end{pmatrix}, \\ &\qquad k \in \bar{\gamma}_d. \end{aligned}$$

Using equalities $1 - f(k)A_+(k)B_+(k) = 0$, $\bar{f}(\bar{k}) = -f(k)$ and $A_+(k)B_+(k) = -A_-(k)B_-(k)$, which follow from the definition of the function $f(k)$, we obtain

$$J^{(4)}(x,t,k) = \begin{cases} \begin{pmatrix} 0 & -f^{-1}(k)\delta^2(k) \\ f(k)\delta^{-2}(k) & 0 \end{pmatrix}, & k \in \gamma_d \\ \begin{pmatrix} 0 & f(k)\delta^2(k) \\ -f^{-1}(k)\delta^{-2}(k) & 0 \end{pmatrix}, & k \in \bar{\gamma}_d. \end{cases}$$

As before, we would like to obtain the RH problem with a jump matrix, which is independent on k. To do so let us use the factorization

$$J^{(4)}(x,t,k) = \begin{pmatrix} \hat{F}_+^{-1}(k) & 0 \\ 0 & \hat{F}_+(k) \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \hat{F}_-(k) & 0 \\ 0 & \hat{F}_-^{-1}(k) \end{pmatrix}$$

which takes place if $\hat{F}_-(k)\hat{F}_+(k) = -\mathrm{i}f(k)\delta^{-2}(k)$ for $k \in \gamma_d$ and $\hat{F}_-(k)\hat{F}_+(k) = \mathrm{i}f^{-1}(k)\delta^{-2}(k)$ for $k \in \bar{\gamma}_d$.

Thus we come to the scalar Riemann–Hilbert problem: find a scalar function $\hat{F}(k)$ such that

- $\hat{F}(k)$ is analytic outside the contour $\gamma_d \cup \bar{\gamma}_d$;
- $\hat{F}(k)$ does not vanish;
- $\hat{F}(k)$ satisfies the jump relation:

$$\hat{F}_-(k)\hat{F}_+(k) = h(k)\delta^{-2}(k), \qquad k \in \gamma_d \cup \bar{\gamma}_d,$$

where

$$h(k) = \begin{cases} -\mathrm{i}f(k), & k \in \gamma_d, \\ \mathrm{i}f^{-1}(k), & k \in \bar{\gamma}_d; \end{cases}$$

- $\hat{F}(k)$ is bounded at infinity.

To find the solution of this RH problem, let us use the function

$$w(k) = \sqrt{(k-E)(k-\bar{E})(k-d)(k-\bar{d})},$$

which together with the jump condition gives

$$\left[\frac{\log \hat{F}(k)}{w(k)}\right]_+ - \left[\frac{\log \hat{F}(k)}{w(k)}\right]_- = \frac{\log[h(k)\delta^{-2}(k,\xi)]}{w_+(k)}, \qquad k \in \gamma_d \cup \bar{\gamma}_d.$$

It is easy to find the unbounded solution in the form

$$\tilde{F}(k) = \exp\left\{\frac{w(k)}{2\pi \mathrm{i}}\int_{\gamma_d\cup\bar{\gamma}_d}\frac{\log[h(s)\delta^{-2}(s,\xi)]}{s-k}\frac{\mathrm{d}s}{w_+(s)}\right\}.$$

Indeed, it has an essential singularity at infinity:

$$\tilde{F}(k) = \tilde{F}_\infty \mathrm{e}^{i\Delta k}\left(1+O\left(\frac{1}{k}\right)\right), \qquad k\to\infty,$$

where

$$\Delta \equiv \Delta(\xi) = \frac{1}{2\pi}\int_{\gamma_d\cup\bar{\gamma}_d}\log[h(s)\delta^{-2}(s,\xi)]\frac{\mathrm{d}s}{w_+(s)}$$

and

$$\tilde{F}_\infty = \exp\left\{\frac{\mathrm{i}}{2\pi}\int_{\gamma_d\cup\bar{\gamma}_d}(s-e_1)\log[h(s)\delta^{-2}(s,\xi)]\frac{\mathrm{d}s}{w_+(s)}\right\}$$

with

$$e_1 = \frac{E+\bar{E}+d+\bar{d}}{2}. \tag{42}$$

To remove this singularity, let us introduce the normalized (zero $\boldsymbol{a}$-period) Abelian integral $\zeta(k)$ of the second kind with the simple pole at infinity:

$$\zeta(k) = \int_E^k \frac{z^2-e_1z+e_0}{w(z)}\,\mathrm{d}z,$$

where e_1 is the same as in (42) and e_0 is defined from the condition

$$e_0 = -\left(\int_d^{\bar{d}}(z^2-e_1z)\frac{\mathrm{d}z}{w(z)}\right)\left(\int_d^{\bar{d}}\frac{\mathrm{d}z}{w(z)}\right)^{-1},$$

i.e.

$$\int_{\boldsymbol{a}}\mathrm{d}\zeta(k) = 0. \tag{43}$$

The large k expansion of $\zeta(k)$ is of the form

$$\zeta(k) = k+\zeta_\infty(\xi)+O(k^{-1}), \qquad k\to\infty,$$

where

$$\zeta_\infty = \int_E^\infty\left[\frac{z^2-e_1z+e_0}{w(z)}-1\right]\mathrm{d}z - E \equiv \frac{1}{2}\left(\int_E^\infty+\int_{\bar{E}}^\infty\right)\left[\frac{z^2-e_1z+e_0}{w(z)}-1\right]\mathrm{d}z - E_1 \tag{44}$$

is a real function of ξ. In the last identity, we have taken into account equation (43) and the absence of residue at infinity in $\mathrm{d}\zeta$. With the same agreement about the choice of the contour of integration as in the case of the Abelian integral $\hat{g}(k)$, we see that the integral $\zeta(k)$ satisfies the similar jump relations:

$$\zeta_+(k)+\zeta_-(k) = 0, \qquad k\in\gamma_d\cup\bar{\gamma}_d;$$
$$\zeta_+(k)-\zeta_-(k) = B_\zeta, \qquad k\in[d,\lambda_-]\cup[\lambda_-,\bar{d}].$$

Here, B_ζ is the $\boldsymbol{b}$-period of the integral $\zeta(k)$:

$$B_\zeta = \int_{\boldsymbol{b}}\mathrm{d}\zeta = 2\int_E^d\frac{z^2-e_1z+e_0}{w(z)}\,\mathrm{d}z = \left(\int_E^d+\int_{\bar{E}}^{\bar{d}}\right)\frac{z^2-e_1z+e_0}{w(z)}\,\mathrm{d}z,$$

where the last equation follows again from (43) and the absence of the residue at infinity. This equation explicitly indicates that $\operatorname{Im} B_\zeta = 0$. Let us now pass from the function $\tilde{F}(k)$ to the function

$$\hat{F}(k) := \tilde{F}(k)\,\mathrm{e}^{-i\Delta\zeta(k)}.$$

This function has no essential singularity at $k = \infty$. Indeed $\hat{F}(\infty) = \exp(\mathrm{i}\hat{\phi}(\xi))$, where

$$\hat{\phi}(\xi) = \frac{1}{2\pi}\int_{\gamma_d\cup\bar{\gamma}_d}(s-e_1)\log[h(s)\delta^{-2}(s,\xi)]\frac{\mathrm{d}s}{w_+(s)} - \Delta\zeta_\infty. \tag{45}$$

Comparing with the function $\tilde{F}(k)$, the function $\hat{F}(k)$ has the same jump relations as $\tilde{F}(k)$ across the arcs γ_d and $\bar{\gamma}_d$, and an extra jump across the broken line $[d,\lambda_-]\cup[\lambda_-,\bar{d}]$. Indeed, we have that

$$\hat{F}_+(k)/\hat{F}_-(k) = \mathrm{e}^{-\mathrm{i}\Delta B_\zeta}.$$

Thus the scalar RH problem for the function $\hat{F}(k)$ has to be modified by adding the extra jump across the broken line $[d,\lambda_-]\cup[\lambda_-,\bar{d}]$. Since

$$J^{(4)}(x,t,k) = \hat{F}_+^{-\sigma_3}(k)J_1^{(\mathrm{mod})}\hat{F}_-^{\sigma_3}(k), \qquad k\in\gamma_d\cup\bar{\gamma}_d,$$

where

$$J_1^{(\mathrm{mod})} = \begin{pmatrix} 0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \tag{46}$$

the next step is as follows:

$$M^{(5)}(x,t,k) = \hat{F}^{\sigma_3}(\infty)M^{(4)}(x,t,k)\hat{F}^{-\sigma_3}(k).$$

Then we have

$$M_-^{(5)}(x,t,k) = M_+^{(5)}(x,t,k)J^{(5)}(x,t,k), \qquad k\in\Sigma^{(5)} := \Sigma^{(4)},$$

where

$$J^{(5)}(x,t,k) = \begin{cases} I, & k\in\mathbb{R},\\ J^{(\mathrm{mod})}, & k\in\gamma_d\cup\bar{\gamma}_d\cup[d,\lambda_-]\cup[\lambda_-,\bar{d}],\\ I+\mathrm{O}(\mathrm{e}^{-\varepsilon t}), & k\in L_j,\quad j=1,2,\ldots,6\end{cases}$$

where

$$J^{(\mathrm{mod})} = \begin{cases} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}tB_g-\mathrm{i}\Delta B_\zeta} & 0\\ 0 & \mathrm{e}^{\mathrm{i}tB_g+\mathrm{i}\Delta B_\zeta}\end{pmatrix}, & k\in[d,\lambda_-]\cup[\lambda_-,\bar{d}],\\ \begin{pmatrix} 0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, & k\in\gamma_d\cup\bar{\gamma}_d.\end{cases} \tag{47}$$

Since $J^{(5)}(x,t,k) = I + O(\mathrm{e}^{-\epsilon t})$ on all arcs $\hat{L}_j$ ($j = 1,2,3,4$) outside small disks around the stationary phase points λ_-,λ_+, we arrive to the one-gap model problem:

$$M_-^{(\mathrm{mod})}(x,t,k) = M_+^{(\mathrm{mod})}(x,t,k)J^{(\mathrm{mod})}, \qquad k\in\gamma_d\cup\bar{\gamma}_d\cup[d,\lambda_-]\cup[\lambda_-,\bar{d}], \tag{48}$$

with the piecewise constant (in k) jump matrix (47) and the asymptotic condition

$$M^{(\mathrm{mod})}(x,t,k) = I + \mathrm{O}\left(\frac{1}{k}\right), \qquad k\to\infty. \tag{49}$$

Note that the jump matrix $J^{(\mathrm{mod})}$ across the broken line $[d,\lambda_-]$, $[\lambda_-,\bar{d}]$ differs from the matrix (41) and coincides with the matrix (46) on $\gamma_d\cup\bar{\gamma}_d$.

Due to the parametrix at the small neighborhoods of the stationary phase points λ_-, λ_+ and, also, end points E and $\bar{E}$, d and $\bar{d}$ we remark the following. The local representation of $\hat{g}(k)$ at the points E and $\bar{E}$ is characterized by a square root type behavior:

$$\hat{g}(k) \sim \hat{g}_0(E,\xi)\sqrt{k-E}, \qquad k \to E; \qquad \hat{g}(k) \sim \bar{\hat{g}}_0(\bar{E},\xi)\sqrt{k-\bar{E}}, \qquad k \to \bar{E}.$$

This means that the associated local model RH problems are solvable in terms of the Bessel functions near the points E and $\bar{E}$ (see [15]). Similarly, the local representation of $g(k)$ at the points d and $\bar{d}$ exhibits a 3/2 root type behavior:

$$\hat{g}(k) \sim B_g + g_2(k-d)^{3/2}, \qquad k \to d; \qquad \hat{g}(k) \sim B_g + \bar{g}_2(k-\bar{d})^{3/2}, \qquad k \to \bar{d}.$$

Hence, in the neighborhoods of the points d and $\bar{d}$, the associated local model RH problems are solvable in terms of the Airy functions (see [16]). The analyses of the parametrix solutions near the λ_-, λ_+ are very similar to the analyses done in [9] and [7]. These are again parabolic cylinder Riemann–Hilbert problems. Skipping the technical details, we have the following asymptotic representation of the function $M^{(5)}(x,t,k)$:

$$M^{(5)}(x,t,k) = \left(I + \mathrm{O}\left(\frac{1}{t^{1/2}}\right)\right) M^{(\mathrm{mod})}(x,t,k),$$

where the error term $\mathrm{O}(t^{-1/2})$ comes from the contribution of the stationary phase points $\lambda_\pm$.

The model problem (48)–(49) with the jump matrix (47) can be solved in elliptic theta functions. For this purpose let us introduce necessary ingredients. Consider elliptic (two band) Riemann surface defined by equation

$$w(k) = \sqrt{(k-E)(k-\bar{E})(k-d(\xi))(k-\bar{d}(\xi))},$$

where the branch points $d(\xi)$ and $\bar{d}(\xi)$ depend on ξ. Let

$$U(k) = \frac{1}{c}\int_E^k \frac{\mathrm{d}z}{w(z)}$$

be the normalized Abelian integral, i.e. its $\boldsymbol{a}$-period is equal to 1. Then

$$c = 2\int_{\bar{d}}^{d} \frac{\mathrm{d}z}{w(z)}, \qquad \tau := \frac{2}{c}\int_E^d \frac{\mathrm{d}z}{w(z)}$$

with $\mathrm{Im}\,\tau > 0$. Furthermore, the following relations are valid

$$U_+(k) + U_-(k) = 0, \qquad k \in \gamma_d;$$

$$U_+(k) + U_-(k) = -1, \qquad k \in \bar{\gamma}_d;$$

$$U_+(k) - U_-(k) = \tau, \qquad k \in [d,\lambda_-] \cup [\lambda_-,\bar{d}].$$

The next ingredient is defined by its asymptotic behavior

$$\tilde{\varkappa}(k) = \left(\frac{(k-\bar{E})(k-\bar{d}(\xi))}{(k-E)(k-d(\xi))}\right)^{1/4} = 1 - \frac{d_2(\xi)+E_2}{2\mathrm{i}k} + \mathrm{O}(k^{-2}), \qquad k \to \infty$$

and cuts along the contour $\gamma_d \cup [d,\lambda_-] \cup [\lambda_-,\bar{d}] \cup \bar{\gamma}_d$. The boundary values of $\tilde{\varkappa}(k)$ from different sides of this contour satisfy the relations

$$\tilde{\varkappa}_-(k) = \mathrm{i}\tilde{\varkappa}_+(k), \qquad k \in \gamma_d \cup \bar{\gamma}_d; \qquad \tilde{\varkappa}_-(k) = -\tilde{\varkappa}_+(k), \qquad k \in [d,\lambda_-] \cup [\lambda_-,\bar{d}].$$

Zeroes of the functions $\tilde{\varkappa}(k) \pm \tilde{\varkappa}^{-1}(k)$ satisfy the equation $\tilde{\varkappa}^4(k) = 1$, which gives

$$E_0 = \frac{Ed(\xi) - \bar{E}\bar{d}(\xi)}{E - \bar{E} + d - \bar{d}(\xi)} = \frac{pd_1(\xi) + ld_2(\xi)}{p + 2\omega d_2(\xi)}.$$

Then, it is easy to find that $\tilde{\varkappa}(E_0) = \mathrm{i}$ and therefore

$$\tilde{\varkappa}(E_0) + \frac{1}{\tilde{\varkappa}(E_0)} = 0, \qquad \tilde{\varkappa}(E_0) - \frac{1}{\tilde{\varkappa}(E_0)} \neq 0.$$

The last ingredient is the theta function with $\operatorname{Im}\tau = \operatorname{Im}\tau(\xi) > 0$:

$$\theta_3(z) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}\tau m^2 + 2\pi\mathrm{i}mz},$$

which is an entire on $z \in \mathbb{C}$ and it has the following properties

$$\theta_3(-z) = \theta_3(z), \quad \theta_3(z+1) = \theta_3(z), \qquad \theta_3(z\pm\tau) = \mathrm{e}^{-\pi\mathrm{i}\tau\mp 2\pi\mathrm{i}z}\theta_3(z).$$

Now introduce the matrix $\Theta(k) = \Theta(t,\xi,k)$ with components

$$\begin{aligned}
\Theta_{11}(t,\xi,k) &= \frac{1}{2}\left[\tilde{\varkappa}(k) + \frac{1}{\tilde{\varkappa}(k)}\right]\frac{\theta_3[U(k) - U(E_0) - \tau/2 - B_g t/2\pi - B_\zeta\Delta/2\pi]}{\theta_3[U(k) - U(E_0) - 1/2 - \tau/2]}\\
\Theta_{12}(t,\xi,k) &= \frac{1}{2}\left[\tilde{\varkappa}(k) - \frac{1}{\tilde{\varkappa}(k)}\right]\frac{\theta_3[U(k) + U(E_0) + \tau/2 + B_g t/2\pi + B_\zeta\Delta/2\pi]}{\theta_3[U(k) + U(E_0) + 1/2 + \tau/2]}\\
\Theta_{21}(t,\xi,k) &= \frac{1}{2}\left[\tilde{\varkappa}(k) - \frac{1}{\tilde{\varkappa}(k)}\right]\frac{\theta_3[U(k) + U(E_0) + \tau/2 - B_g t/2\pi - B_\zeta\Delta/2\pi]}{\theta_3[U(k) + U(E_0) + 1/2 + \tau/2]}\\
\Theta_{22}(t,\xi,k) &= \frac{1}{2}\left[\tilde{\varkappa}(k) + \frac{1}{\tilde{\varkappa}(k)}\right]\frac{\theta_3[U(k) - U(E_0) - \tau/2 + B_g t/2\pi + B_\zeta\Delta/2\pi]}{\theta_3[U(k) - U(E_0) - 1/2 - \tau/2]},
\end{aligned} \tag{50}$$

where E_0 is the zero of the function $\tilde{\varkappa}(k) + \tilde{\varkappa}^{-1}(k)$. We consider this matrix as a function on the upper (first) sheet of the Riemann surface cut along the contour $\gamma_d \cup [d,\lambda_-] \cup [\lambda_-,\bar{d}] \cup \bar{\gamma}_d$. Then such a restriction of the matrix $\Theta(t,\xi,k)$ is analytic and bounded in $k \in \mathbb{C}\setminus\{\gamma_d \cup \bar{\gamma}_d \cup [d,\lambda_-] \cup [\lambda_-,\bar{d}]\}$, because due to the theory of Abelian theta function, $\Theta_{12}(k)$ and $\Theta_{21}(k)$ have poles on the lower sheet of the Riemann surface and they are analytic and bounded on the upper sheet. The poles of $\Theta_{11}(k)$ and $\Theta_{22}(k)$ on the upper sheet annihilate by zero E_0 of the factor $\tilde{\varkappa}(k) + \tilde{\varkappa}^{-1}(k)$. The restriction of the matrix $\Theta(t,\xi,k)$ on the upper sheet satisfies the jump conditions (47)–(48) of the one-gap model RH problem. Indeed, this follows immediately from the above-mentioned properties of the function $\tilde{\varkappa}(k)$, the theta-function $\theta_3(z)$ and the Abelian integral $U(k)$. Thus we have that $[\Theta_{11}]_-(t,\xi,k) = \mathrm{i}[\Theta_{12}]_+(t,\xi,k)$ for $k \in \gamma_d \cup \bar{\gamma}_d$ and $[\Theta_{11}]_-(t,\xi,k) = \mathrm{e}^{\mathrm{i}\pi}[\Theta_{11}]_+(t,\xi,k)\,\mathrm{e}^{-\mathrm{i}B_g t - \mathrm{i}B_\zeta\Delta - \mathrm{i}\pi}$ for $k \in [d,\lambda_-] \cup [\lambda_-,\bar{d}]$ and so on. Since $\Theta_{21}(t,\xi,\infty) = \Theta_{12}(t,\xi,\infty) = 0$ then the matrix $\Theta^{-1}(t,\xi,\infty)$ does exist and the solution of the one-gap model RH problem normalized at infinity (49) is given by equation

$$M^{(\mathrm{mod})}(x,t,k) = \Theta^{-1}(t,\xi,\infty)\Theta(t,\xi,k).$$

Finally we have the following chain of transformations of the RH problem:

$$\begin{aligned}
&M(x,t,k) = M^{(1)}(x,t,k)[G^{(1)}(k)]^{-1},\\
&M^{(1)}(x,t,k) = \mathrm{e}^{\mathrm{i}t\hat{g}_\infty(\xi)\sigma_3} M^{(2)}(x,t,k)\,\mathrm{e}^{\mathrm{i}t[\theta(k)-\hat{g}(k)]\sigma_3},\\
&M^{(2)}(x,t,k) = M^{(3)}(x,t,k)\delta^{\sigma_3}(k),\\
&M^{(3)}(x,t,k) = M^{(4)}(x,t,k)[G^{(2)}(k)]^{-1},\\
&M^{(4)}(x,t,k) = \hat{F}^{-\sigma_3}(\infty)M^{(5)}(x,t,k)\hat{F}^{\sigma_3}(k),\\
&M^{(5)}(x,t,k) = M^{(\mathrm{mod})}(x,t,k)(I + O(t^{-1/2})).
\end{aligned}$$

Let us remind that any matrix $M^{(j)}(x,t,k)$ $(j = 1,2,\ldots)$ defines the same functions $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$ since all bordering matrices are diagonal at the point $k = \infty$ and $k = 0$. Due to the theorem 2.1 $\hat{Q}(x,t) = -M(x,t,0)\sigma_3 M^{-1}(x,t,0)$. Hence

$$\begin{aligned}
\hat{Q}(x,t) = \begin{pmatrix} \nu(x,t) & \mathrm{i}\mu(x,t)\\ -\mathrm{i}\mu(x,t) & -\nu(x,t)\end{pmatrix} &= -\mathrm{e}^{\mathrm{i}t\hat{g}_\infty(\xi)\sigma_3}\hat{F}^{-\sigma_3}(\infty)\Theta^{-1}(t,\xi,\infty)\\
&\times \Theta(t,\xi,0)\sigma_3\Theta^{-1}(t,\xi,0)\Theta(t,\xi,\infty)\hat{F}^{\sigma_3}(\infty)\,\mathrm{e}^{-\mathrm{i}t\hat{g}_\infty(\xi)\sigma_3} + O(t^{-1/2}).
\end{aligned}$$

Since $\hat{F}(\infty) = \mathrm{e}^{\mathrm{i}\hat{\phi}(\xi)}$ and

$$\Theta^{-1}(t,\xi,\infty)\Theta(t,\xi,0) = \begin{pmatrix} \dfrac{\Theta_{11}(t,\xi,0)}{\Theta_{11}(t,\xi,\infty)} & \dfrac{\Theta_{12}(t,\xi,0)}{\Theta_{11}(t,\xi,\infty)} \\[2ex] \dfrac{\Theta_{21}(t,\xi,0)}{\Theta_{22}(t,\xi,\infty)} & \dfrac{\Theta_{22}(t,\xi,0)}{\Theta_{22}(t,\xi,\infty)} \end{pmatrix},$$

we find

$$\nu(x,t) = -1 + 2\frac{\Theta_{11}(t,\xi,0)\Theta_{22}(t,\xi,0)}{\Theta_{11}(t,\xi,\infty)\Theta_{22}(t,\xi,\infty)}$$

and

$$\mu(x,t) = 2\mathrm{i}\frac{\Theta_{11}(t,\xi,0)\Theta_{12}(t,\xi,0)}{\Theta_{11}^2(t,\xi,\infty)}\exp\left[2\mathrm{i}t\hat{g}_\infty(\xi) - 2\mathrm{i}\hat{\phi}(\xi)\right],$$

where $\Theta_{ij}(t,\xi,k)$ are given in (50), $\hat{g}_\infty(\xi)$ in (34) and $\hat{\phi}(\xi)$ in (45).

Remark 3.2. If we take into account that

$$\bar{\mu}(x,t) = 2\mathrm{i}\frac{\Theta_{22}(t,\xi,0)\Theta_{21}(t,\xi,0)}{\Theta_{22}^2(t,\xi,\infty)}\exp\left[-2\mathrm{i}t\hat{g}_\infty(\xi) + 2\mathrm{i}\hat{\phi}(\xi)\right]$$

and identity (determinant relation)

$$\Theta_{11}(t,\xi,0)\Theta_{22}(t,\xi,0) - \Theta_{12}(t,\xi,0)\Theta_{21}(t,\xi,0) \equiv \Theta_{11}(t,\xi,\infty)\Theta_{22}(t,\xi,\infty)$$

then we easily find that adopted normalization condition for the solution of the IBV problem

$$\nu^2(x,t) + |\mu(x,t)|^2 \equiv 1$$

is fulfilled.

Again by the theorem 2.1 $q(x,t) = 2\mathrm{i}m_{12}(x,t)$. Taking into account the chain of our transformations, we have

$$\begin{aligned} q(x,t) &= 2\mathrm{i}m_{12}(x,t) = 2\mathrm{i}m_{12}^{(1)}(x,t) = 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}t\hat{g}_\infty(\xi)}m_{12}^{(2)}(x,t) \\ &= 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}t\hat{g}_\infty(\xi)}m_{12}^{(3)}(x,t) + \mathrm{O}(t^{-1/2}) \\ &= 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}t\hat{g}_\infty(\xi)}m_{12}^{(4)}(x,t) + \mathrm{O}(t^{-1/2}) \\ &= 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}t\hat{g}_\infty(\xi)}m_{12}^{(5)}(x,t)\hat{F}^{-2}(\infty) + \mathrm{O}(t^{-1/2}) \\ &= 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}t\hat{g}_\infty(\xi)}m_{12}^{(\mathrm{mod})}(x,t)\hat{F}^{-2}(\infty) + \mathrm{O}(t^{-1/2}) \\ &= 2\mathrm{i}\,\mathrm{e}^{2\mathrm{i}t\hat{g}_\infty(\xi)}\frac{\Theta_{12}(t,\xi,\infty)}{\Theta_{11}(t,\xi,\infty)}\hat{F}^{-2}(\infty) + \mathrm{O}(t^{-1/2}) \\ &= -[d_2(\xi) + E_2]\frac{\theta_3[B_g t/2\pi + B_\zeta\Delta/2\pi + V_+]}{\theta_3[B_g t/2\pi + B_\zeta\Delta/2\pi + V_-]}\frac{\theta_3[V_- + 1/2]}{\theta_3[V_+ + 1/2]}\exp[2\mathrm{i}t\hat{g}_\infty(\xi) - 2\mathrm{i}\hat{\phi}(\xi)] \\ &\quad + \mathrm{O}(t^{-1/2}), \end{aligned}$$

with $V_\pm = \pm U(\infty) + U(E_0) + \tau/2$ and $U(\infty) = U(\infty,\xi)$, $U(E_0) = U(E_0,\xi)$, $\tau = \tau(\xi)$.

The above asymptotic formulas are valid for $\omega_0^2 t < x < \omega^2 t$ because when $\xi = \xi_0$ ($x = \omega_0^2 t$, $\omega_0^2 = 1/4\xi_0^2$) the function $d_2(\xi_0)$ vanishes and the genus of the elliptic Riemann surface degenerates to the trivial one.

Theorem 3.2. *The solution of the IBV problems (1)–(4) for $t \to \infty$ in the region $\omega_0^2 t < x < \omega^2 t$ takes the form of the modulated elliptic wave*

$$q(x,t) = 2\mathrm{i}\frac{\Theta_{12}(t,\xi,\infty)}{\Theta_{11}(t,\xi,\infty)}\exp[2\mathrm{i}t\hat{g}_\infty(\xi) - 2\mathrm{i}\hat{\phi}(\xi)] + \mathrm{O}(t^{-1/2}),$$

$$\nu(x,t) = -1 + 2\frac{\Theta_{11}(t,\xi,0)\Theta_{22}(t,\xi,0)}{\Theta_{11}(t,\xi,\infty)\Theta_{22}(t,\xi,\infty)} + \mathrm{O}(t^{-1/2}),$$

$$\mu(x,t) = 2\mathrm{i}\frac{\Theta_{11}(t,\xi,0)\Theta_{12}(t,\xi,0)}{\Theta_{11}^2(t,\xi,\infty)}\exp[2\mathrm{i}t\hat{g}_\infty(\xi) - 2\mathrm{i}\hat{\phi}(\xi)] + \mathrm{O}(t^{-1/2}),$$

where

$$\hat{g}_\infty(\xi) = \left(\int_E^\infty + \int_{\bar{E}}^\infty\right) \times \left[\left(1 - \frac{\lambda_-(\xi)}{z}\right)\left(1 - \frac{\lambda_+(\xi)}{z}\right)\sqrt{\frac{(z-d(\xi))(z-\bar{d}(\xi))}{(z-E)(z-\bar{E})}} - 1\right]\frac{\mathrm{d}z}{8\xi^2} - \frac{l}{8\omega\xi^2}$$

is the regularized meaning of the phase function $\hat{g}(k) = \hat{g}(k,\xi)$ at infinity. The phase shift $\hat{\phi}(\xi)$ is defined by

$$\hat{\phi}(\xi) = \frac{1}{2\pi}\int_{\gamma_d\cup\bar{\gamma}_d}(k - e_1 - \zeta_\infty)\log[h(k)\delta^{-2}(k,\xi)]\frac{\mathrm{d}k}{w_+(k)},$$

$$h(k) = \begin{cases} -\mathrm{i}f(k), & k \in \gamma_d \\ \mathrm{i}f^{-1}(k), & k \in \bar{\gamma}_d \end{cases},$$

$$\delta(k) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\lambda_-(\xi)}^{\lambda_+(\xi)}\frac{\log(1-\rho^2(s))\,\mathrm{d}s}{s-k}\right\}, \qquad k \in \mathbb{C}\setminus[\lambda_-(\xi),\lambda_+(\xi)], \qquad |E| \leqslant \xi \leqslant \xi_0,$$

where $f(k)$, $\rho(k)$ are spectral functions, and $e_1 = e_1(\xi)$, $\zeta_\infty = \zeta_\infty(\xi)$, $d(\xi)$, $\lambda_\pm(\xi)$ are defined by equations (42), (44), (37), (38), (39) and (40).

Remark 3.3. Since $g_\infty(\xi_0) = \hat{g}_\infty(\xi_0) = \omega/2 - l/8\omega\xi_0^2$, $\phi(\xi_0) = \hat{\phi}(\xi_0)$, $\mathrm{Im}\,d(\xi_0) = 0$ and theta-function $\theta_3(*)|_{\xi=\xi_0} \equiv 1$, where $\xi_0 = \frac{1}{2\omega_0}$, we have that elliptic wave matches the plane wave at $\xi = \xi_0$.

3.3. Vanishing dispersive asymptotics ($x > \omega^2 t$)

To study the asymptotic behavior of the Riemann–Hilbert problem RH_{xt} in the region $x > \omega^2 t$ we have used well-known techniques from [8, 9, 15]. The large time asymptotics of the solution in this region is defined by the phase function $\theta(k) = \frac{1}{4}\left(\frac{1}{k} + \frac{k}{\xi^2}\right)$, where $\xi^2 = t/4x$. Indeed, the stationary points of the phase function $\theta(k)$ are real and equal to $\pm\xi$. We have

$$\mathrm{Im}\,\theta(k) = |k|^2 - \xi^2 4|k|^2\xi^2\mathrm{Im}\,k.$$

Therefore, $\mathrm{Im}\,\theta(k) > 0$ ($\mathrm{Im}\,\theta(k) < 0$) for k lying in the lower (upper) half-disk and out of the upper (lower) half-disk defined by the circle $|k|^2 = \xi^2$ (figure 3). For $\xi^2 < |E|^2 = 1/4\omega^2$ (that is, for $x > \omega^2 t$) and for $k \in \gamma \cup \bar{\gamma}$, the jump matrix $J^{(1)}(x,t,k)$ tends to the identity matrix as $t \to \infty$. Hence the contour $\gamma \cup \bar{\gamma}$ does not contribute to the main term of the asymptotics, which equals zero. The next term of the asymptotics, defined by the stationary points $\pm\xi$, has the order $O(t^{-1/2})$. This vanishing term of the asymptotics of the solution was obtained in [7].

Theorem 3.3. *The solution of the IBV problem (1)–(4) for $t \to \infty$ in the region $x > \omega^2 t$ takes the form of vanishing self-similar wave:*

$$q(x,t) = 2\sqrt{\frac{\xi^3\eta(\xi)}{t}}\exp\{2\mathrm{i}\sqrt{xt} - \mathrm{i}\eta(\xi)\log\sqrt{xt} + i\varphi(\xi)\}$$

$$+\,2\sqrt{\frac{\xi^3\eta(-\xi)}{t}}\exp\{-2\mathrm{i}\sqrt{xt} + \mathrm{i}\eta(-\xi)\log\sqrt{xt} + i\varphi(-\xi)\} + o(t^{-1/2}), \qquad t \to \infty,$$

where the functions $\eta(k)$ and $\varphi(k)$ are given by the equations

$$\eta(k) = \frac{1}{2\pi}\log(1-\rho^2(k)), \qquad \xi^2 = \frac{t}{4x},$$

$$\varphi(k) = \frac{\pi}{4} - 3\eta(k)\log 2 - \arg\rho(k) - \arg\Gamma(-\mathrm{i}\eta(k)) + \frac{1}{\pi}\int_{-\xi}^{\xi}\log|s-k|d\log[1-\rho^2(s)].$$

Here $\Gamma(-\mathrm{i}\eta(k))$ is the Euler gamma-function and $\rho(k) = \frac{\varkappa^2(k)-1}{\varkappa^2(k)+1}$.

Remark 3.4. In the paper [12], the asymptotic behavior of the solution of the problem (1), (2), (6) was studied in a neighborhood of the leading edge $x = \omega^2 t$ in terms of asymptotic solitons. The problem of matching of the elliptic wave with the asymptotic solitons and these solitons with the vanishing self-similar wave is much more complicated and will be considered elsewhere.

Acknowledgment

The authors thank D G Shepelsky for comments and remarks.

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